PLURICANONICAL MAPS OF STABLE LOG SURFACES

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Abstract. Stable surfaces and their log analogues are the type of varieties naturally occuring as boundary points in moduli spaces. We extend classical results of Kodaira and Bombieri to this more general setting: if (X, Δ) is a stable log surface with reduced boundary (possibly empty) and I is its global index, then $4I(K_X + \Delta)$ is base-point-free and $8I(K_X + \Delta)$ is very ample.

These bounds can be improved under further assumptions on the singularities or invariants; for example, $5(K_X + \Delta)$ is very ample if (X, Δ) has semi-canonical singularities.

Contents

1. Introduction

It is a general fact that moduli spaces of *nice* objects in algebraic geometry, say smooth varieties, are often non-compact. But usually there is a modular compactification where the boundary points correspond to related, but more complicated objects.

Such a modular compactification has been known for the moduli space $\mathcal{M}_g$ of smooth curves of genus g for a long time and in [KSB88] Kollár and Shepherd-Barron made the first step towards the construction of a modular compactification $\overline{\mathfrak{M}}$ for the moduli space $\mathfrak{M}$ of surfaces of general type. Even though the actual construction of the moduli space was delayed for several decades because of formidable technical obstacles to be overcome, it was clear from the beginning that the objects parametrised by $\overline{\mathfrak{M}}$ should be surfaces with semi-log-canonical singularities and ample canonical divisor, for short *stable surfaces*.

A more general version also incorporates the possibility of a (reduced) boundary divisor (see Section 2.2 for the precise definitions); this is the higher dimensional analogue of pointed stable curves and was worked out by Alexeev [Ale96, Ale06].

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In recent years, several components of the moduli space of stable varieties or pairs have been investigated in detail[1]. On the other hand, many of the standard tools to study, for example, smooth surfaces of general type are not yet available for stable surfaces. In this paper, we make the first steps in the understanding of pluricanonical maps of surfaces.

Pluricanonical maps are one of the main tools in the study of smooth surfaces of general type and their canonical models. They have been an active subject of research ever since Bombieri's seminal paper [Bom73]. Recall that the m-canonical map of a variety X is the rational map $\varphi_m\colon X \dashrightarrow \mathbb{P}^N$ associated to the linear system $|mK_X|$. Then the roughest version of Bombieri's results says that on a surface of general type with canonical singularities and ample canonical divisor φ_m is an embedding for $m \geq 5$; it had been proved earlier by Kodaira that φ_m is a morphism for $m \geq 4$ [Kod68]. These results are sharp but can be much refined and we refer to [BHPV04, Sect. VII] or the recent survey [BCP06] for more information.

The singularities of a stable surface are much worse than canonical singularities: in general they are non-normal, not Gorenstein and not (semi-)rational. Thus many of the techniques which one can use to prove Bombieri-type theorems do not carry over directly. The following theorem is proved in Section 4 by applying a Reider-type result of Kawachi's on the normalisation combined with a detailed analysis of the non-normal locus.

Theorem 4.1 — *Let (X, Δ) be a connected stable log surface with global index I.*[2]

(*i*) *The line bundle $\omega_X(\Delta)^{[mI]}$ is base-point-free for $m \geq 4$.*

(*ii*) *The line bundle $\omega_X(\Delta)^{[mI]}$ is base-point-free for $m \geq 3$ if one of the following holds:*

  (a) *$I \geq 2$.*

  (b) *There is no irreducible component $\bar{X}_i$ of the normalisation such that $\left(\pi^*(K_X + \Delta)|_{\bar{X}_i}\right)^2 = 1$ and the union of Δ and the non-normal locus is a nodal curve.*

  (c) *X is normal and we do not have $I = (K_X + \Delta)^2 = 1$.*

For normal stable surfaces without boundary this recovers [KM98a, Cor. 3, Cor. 4].

Our results on pluri-log-canonical embeddings are somewhat more involved. We follow an approach due to Catanese and Franciosi [CF96], later refined in collaboration with Hulek and Reid [CFHR99]: for every subscheme of length two find a pluri-log-canonical curve containing it and then prove that this curve is embedded by $|mI(K_X + \Delta)|$. Without further assumptions on singularities and invariants we get:

General bounds (Theorem 5.1) — *Let (X, Δ) be a connected stable log surface of global index I.*

(*i*) *The line bundle $\omega_X(\Delta)^{[mI]}$ is very ample for $m \geq 8$.*

(*ii*) *The line bundle $\omega_X(\Delta)^{[mI]}$ defines a birational morphism for $m \geq 6$.*

(*iii*) *The line bundle $\omega_X(\Delta)^{[mI]}$ is very ample for $m \geq 6$ if $I \geq 2$.*

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[1] The following is a probably incomplete list of results in this direction: [Has99, Lee00, Hac04, vO05, vO06b, vO06a, AP09, HKT09, Rol10, Liu12, Laz12, BHPS12, Pat12].

[2] Multiplication with the index is clearly necessary since φ_m cannot be a morphism if $m(K_X+\Delta)$ is not a Cartier divisor. However, it does make sense to ask which is the first pluri-log-canonical map to be birational. In the normal case Langer has given an explicit but still unrealistically large bound [Lan01, Sect. 5].

We do not believe all of these bounds to be sharp. The main obstacles in our proof are the extra contributions from the worse than canonical singularities and the fact that curves containing irreducible components of the non-normal locus and the boundary do not behave well under normalisation. We explain this more in detail in Section 5.1, see also Remark 5.8.

Under additional assumptions we can improve the bounds obtained above. In particular, if X is semi-canonical then we obtain the same bound as in the classical case.

Bounds for milder singularities (Theorem 5.2) — *Let (X, Δ) be a connected stable log surface of global index I.*

(*i*) *The line bundle $\omega_X(\Delta)^{[mI]}$ is very ample for $m \geq 7$ if one of the following holds:*
   (a) *There is no irreducible component $\bar{X}_i$ of the normalisation such that $\left(\pi^*(K_X + \Delta)|_{\bar{X}_i}\right)^2 = 1$ and the union of Δ and the non-normal locus is a nodal curve.*
   (b) *X is normal and not $(K_X + \Delta)^2 = 1$.*

(*ii*) *The line bundle $\omega_X(\Delta)^{[mI]}$ is very ample for $m \geq 6$ if the normalisation $\bar{X}$ is smooth along the conductor divisor and has at most canonical singularities elsewhere.*

(*iii*) *The line bundle $\omega_X(\Delta)^{[mI]}$ is very ample for $m \geq 5$ if $D \cup \Delta$ is a nodal curve, $\bar{X}$ is smooth along the conductor divisor, and $X \setminus D$ has at most canonical singularities.*

In particular these conditions are satisfied, if (X, Δ) has semi-canonical singularities.

For a connected stable surface X with canonical singularities the bi-canonical map is a morphism as soon as $K_X^2 \geq 5$ and the tri-canonical map is an embedding as soon as $K_X^2 \geq 6$ (see [Cat87]). Such behaviour cannot be expected for stable surfaces: in Example 7.2 we construct an irreducible, Gorenstein stable surface with K_X^2 arbitrarily large but the bi-canonical map not a morphism and the tri-canonical map not an embedding.

A natural extension of the aforementioned results is the study of the (index)-canonical ring. We do not engage in a detailed study but only state the results that follow by standard methods from Theorem 4.1.

Theorem 6.1 — *Let (X, Δ) be a stable log surface of index I. Then the index-log-canonical ring,*

$$R^{(I)} = \bigoplus_{m \in \mathbb{Z}_{\geq 0}} H^0(X, \omega_X(\Delta)^{[mI]}),$$

is generated in degree at most 13 and in degree 10 under the same assumptions as in Theorem 4.1(ii)

All the results should only be regarded as a first step towards a precise understanding of pluri-log-canonical maps and pluri-log-canonical rings of stable surfaces.

Our method relies on the rough classification of semi-log-canonical singularities and thus does not generalise to higher dimensions at the moment.

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1.1. **Notations and conventions.** We work exclusively with schemes of finite type over the complex numbers.

- The singular locus of a scheme X will be denoted by X_{sing}.
- A surface is a reduced, projective scheme of pure dimension two but not necessarily irreducible or connected.
- A curve is a purely 1-dimensional scheme that is Cohen–Macaulay. A curve is not assumed to be reduced, irreducible or connected. For a point $p \in C$ we denote by $\mu_p(C)$ its multiplicity.
- For a curve C (possibly non-connected) we define the arithmetic genus to be $p_a(C) = 1 - \chi(\mathcal{O}_C)$.
- Let X be a scheme which is Gorenstein in codimension 1 and S_2. For a sheaf $\mathcal{F}$ on X we denote by $\mathcal{F}^{[m]} = \mathscr{H}om_{\mathcal{O}_X}(\mathscr{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{O}_X), \mathcal{O}_X)$ the reflexive powers.

  We switch back and forth between multiples of the canonical divisor, mK_X, and reflexive powers of the canonical sheaf $\omega_X^{[m]}$.
- We call a $\mathbb{Q}$-Weil divisor reduced if all non-zero coefficients are equal to 1.

Some further notation on demi-normal schemes or semi-log-canonical pairs will be fixed in Notation 2.4

2. Set-up

We will now recall the definition of semi-log-canonical singularity and stable log surface and some of their properties. Our main references are [Kol12, KM98b] but for simplicity we stick to dimension 2 from the beginning.

2.1. **Log-canonical pairs in dimension two.** Let X be a normal surface and $\pi\colon Y \to X$ be a birational morphism. Using the negative definiteness of the intersection form on exceptional divisors Mumford defined the pullback for all Weil divisors in X and consequently a $\mathbb{Q}$-valued intersection pairing on Weil-divisors on a normal surface (see [Sak84, Sect. 1]).

Let Δ be a (possibly empty or non-effective) $\mathbb{Q}$-Weil-divisor on X and $\pi\colon Y \to X$ a resolution of singularities such that the union of the strict transform of Δ and the exceptional divisor is a simple normal crossing divisor. Denote by E_i the exceptional divisors. Then there exist unique rational numbers a_i, called discrepancies, such that

$$\pi^*(K_X + \Delta) \sim_{\mathbb{Q}} K_Y + (\pi^{-1})_*\Delta - \sum_i a_i E_i.$$

Definition 2.1 — Let X be a normal surface and Δ a reduced Weil-divisor on X. The pair (X, Δ) is called log-canonical (resp. canonical) if $K_X + \Delta$ is $\mathbb{Q}$-Cartier and, in the notation above, $a_i \geq -1$ (resp. $a_i \geq 0$) for all i.

Remark 2.2 — There are several related definitions of singularities (terminal, klt, dlt, ...), which however will not play a role for us.

We recall the following facts from the classification of surface singularities (see [KM98b, Sct. 4.1]): if $x \in X$ is a normal surface singularity without boundary then x is

- a smooth point, or
- an ordinary double point of type A_n, D_n or E_n (canonical), or
- a finite quotient singularity, or
- a (quotient of a) simple elliptic or cusp singularity (log-canonical).

If $x \in \Delta \subset X$ is a log-canonical singularity then $x \in X$ is a finite quotient singularity, Δ is either smooth or has an ordinary double point at x. The extended resolution graphs, basically specifying the tangent direction of Δ are classified in [KM98b, Thm. 4.15].

2.2. **Non-normal surfaces.** We now define stable log surfaces and explain some parts of [Kol12, Ch. 5]. For the sake of simplicity we restrict to dimension two from the beginning.

2.2.1. *Demi-normal surfaces.*

Definition 2.3 (Kollár) — A 2-dimensional scheme X is called demi-normal if it satisfies Serre's condition S_2 and at each point of codimension 1, X is either regular or has an ordinary double point.

We now fix some notation that will be used in the rest of the article.

Notation 2.4 — Let X be a demi-normal surface with normalisation $\pi\colon \bar{X} \to X$. The conductor ideal $\mathrm{cond}_X = \mathscr{H}om_{\mathcal{O}_X}(\pi_*\mathcal{O}_{\bar{X}}, \mathcal{O}_X)$ is an ideal sheaf in both $\mathcal{O}_X$ and $\mathcal{O}_{\bar{X}}$ and as such defines subschemes

$$D \subset X \text{ and } \bar{D} \subset \bar{X},$$

both reduced and of pure codimension 1 [Kol12, Sect. 5.1]; we often refer to D as the non-normal locus of X.

The map $\pi\colon \bar{D} \to D$ is generically a double cover and if we normalise the non-normal locus and its preimage we get a diagram

$$\begin{tikzcd} \bar{X} \arrow[d, "\pi"] & \bar{D} \arrow[l, hook'] \arrow[d, "\pi"] & \bar{D}^\nu \arrow[l, "\bar{\nu}"'] \arrow[d, "\pi"] \\ X & D \arrow[l, hook'] & D^\nu \arrow[l, "\nu"'] \end{tikzcd}$$

where $\pi\colon \bar{D}^\nu \to D^\nu$ is a double cover. Thus there is an involution[3]

$$\tau\colon \bar{D}^\nu \to \bar{D}^\nu \text{ such that } D^\nu = \bar{D}^\nu/\tau.$$

Weil-divisors containing a component of the non-normal locus do not behave very well, so in order not to repeat ourselves over and over again we encode the property in a definition, which we will also need for pairs.

Definition 2.5 — A log surface is a demi-normal surface X together with a reduced Weil-divisor Δ, called boundary divisor, such that the support of Δ does not contain any irreducible component of the conductor divisor D.

A $\mathbb{Q}$-Weil-divisor C on a log surface (X, Δ) is called *well-behaved on* (X, Δ) if its support does not contain any irreducible component of $D \cup \Delta$.

[3]It is an important point here that, in general, τ does not induce an involution on $\bar{D}$.

A demi-normal surface X is Gorenstein in codimension 1 and S_2 and we can use the theory of generalised divisors from [Har92]. In particular, for every integral Weil-divisor A there exists an associated reflexive sheaf $\mathcal{O}_X(A)$ and if A is well-behaved on X in the sense of Definition 2.5, then it is almost Cartier. Almost Cartier divisors form a group under addition. The corresponding operation on the level of reflexive sheaves is first taking the tensor product and then the double dual.

Let ω_X be the dualising sheaf which coincides with the pushforward of the canonical bundle on the Gorenstein locus. A canonical divisor K_X is an integral Weil divisor such that $\mathcal{O}_X(K_X) = \omega_X$. By a local computation we have $(\pi^*\omega_X)^{[1]} \cong \omega_Y(\bar{D})$.

Definition 2.6 — Let X be a demi-normal surface.

(i) We define a $\mathbb{Q}$-valued intersection pairing for well-behaved Weil-divisors in the following way: let A, B be well-behaved Weil-divisors on X and $\bar{A}$, $\bar{B}$ their strict transforms the normalisation $\bar{X}$. Then the intersection number is

$$AB := \bar{A}\bar{B}$$

where we use Mumfords intersection pairing for normal surfaces. (see e.g. [Sak84]).

(ii) For a $\mathbb{Q}$-Cartier Weil divisor F and a curve B on X we denote by

$$\deg F|_B := \frac{1}{m} \deg \mathcal{O}_X(mF)|_B,$$

where m is a positive integer such that mF is Cartier. such that $f^*A - (f^{-1})_*A$ is f-exceptional and $f^*AE = 0$ for all f-exceptional curves E.

Remark 2.7 — The intersection form defined in this way has some unexpected properties: For example, if A and B are contained in different irreducible components of X then their intersection number is zero even if they intersect in the non-normal locus.

On the other hand, if in (ii) both F and B are well-behaved then $\deg F|_B = FB$.

Remark 2.8 — If A is an almost Cartier divisor and B is a well-behaved curve in X then the restriction $\mathcal{O}_X(A)|_B = \mathcal{O}_X(A) \otimes \mathcal{O}_B$ may have torsion; we will use the notation $\mathcal{O}_B(A) := (\mathcal{O}_Y(A) \otimes \mathcal{O}_B)/\mathrm{tor}$, where tor denotes the torsion part of $\mathcal{O}_Y(A) \otimes \mathcal{O}_B$.

Note that if A is $\mathbb{Q}$-Cartier but not Cartier then our definition of degree $\deg A|_B = AB$ does not satisfy the usual Riemann–Roch formula for curves, that is, in general $\chi(\mathcal{O}_B(A)) \neq \deg A|_B + \chi(\mathcal{O}_B)$ (see Proposition A.5). Thus our definition of degree agrees with the definition in [CFHR99] only for Cartier divisors.

2.2.2. *Semi-log-canonical and stable log surfaces.* Philosophically, semi-log-canonical singularities are the analogue of log-canonical singularities for demi-normal schemes.

Definition 2.9 — Let (X, Δ) be a log surface and let $\bar{\Delta} \subset \bar{X}$ be the strict transform of Δ in the normalisation. The log surface (X, Δ) is said to have semi-log-canonical (semi-canonical) singularities if $K_X + \Delta$ is $\mathbb{Q}$-Cartier and the pair $(\bar{X}, \bar{D} + \bar{\Delta})$ is log-canonical (canonical). The global index I of (X, Δ) is the smallest positive integer such that $I(K_X + \Delta)$ is Cartier.

The log surface (X, Δ) is called stable if it has semi-log-canonical singularities and $K_X + \Delta$ is ample. If the boundary is empty then we simply call X a stable surface.

Semi-log-canonical will often be abbreviated as slc. A rough classification of slc surface singularities without boundary is already given in [KSB88]; an additional

list of slc surface singularities with reduced boundary can be deduced from their normalisations ([Kol12, Theorem 4.15], [Kol10, 17]) and the gluing process explained after Theorem 2.13. Equations for all two-dimensional slc hypersurfaces can be found in [LR12b].

The definitions can be extended to include a fractional boundary divisor but we do not need this more general setting.

If one is only interested in surfaces of general type and their degenerations one can assume $\Delta = 0$ throughout. However, stable log surfaces have found several interesting applications to moduli of plane curves, abelian varieties, K3-surfaces and del Pezzo surfaces[4].

Definition 2.10 — Let (X, Δ) be a log surface and B a well-behaved reduced curve on X. Let B^ν be the normalization of B. Suppose $\omega_X(\Delta + B)^{[m]}$ is a line bundle for some positive integer m. Then the different $\mathrm{Diff}_{B^\nu}(\Delta)$ is a $\mathbb{Q}$-divisor on B^ν such that

$$\omega_X(\Delta + B)^{[m]}|_B \cong \omega_{B^\nu}^{[m]}(m\mathrm{Diff}_{B^\nu}(\Delta)).$$

Though surfaces with slc singularities can be recovered from their normalisations together with some additional data (see Theorem 2.13), it is sometimes useful to consider their semi-resolution, first considered by Kollár and Shepherd-Barronc and independently by van Straten under a different name.

Definition 2.11 ([KSB88], [Kol12]) — A surface Y is called semi-smooth if every point of Y is either smooth or double normal crossing or a pinch point[5].

A well-behaved smooth rational curve E on a semi-smooth surface Y is called a (-1)-curve, if $E^2 = -1$ and $\deg K_Y|_E \leq 0$.

A morphism of demi-normal surfaces $f\colon Y \to X$ is called a semi-smooth resolution if the following conditions are satisfied:

(i) Y is semi-smooth;
(ii) there is a semi-smooth open subscheme U of X such that the codimension of $X \setminus U$ is two and f is an isomorphism over U;
(iii) f maps the singular locus of Y birationally onto the non-normal locus of X.

A semi-resolution is called minimal if it does not contract (-1)-curves.

Theorem 2.12 ([vS87], [Kol12, Thm. 9.53]) — *Let X be a demi-normal surface. Then X has a unique minimal semi-resolution.*

The possible configurations of exceptional divisors on the semi-resolution of an slc point will be discussed in Section A.4.

The following result was proved by Kollár in any dimension:

Theorem 2.13 (Kollár) — *The normalisation procedure described in Notation 2.4 gives a bijection of sets*

$$\left\{\begin{array}{c}\textit{stable log}\\ \textit{surface } (X,\Delta)\end{array}\right\} \leftrightarrow \left\{(\bar X, \bar D, \tau)\ \middle|\ \begin{array}{l}(\bar X, \bar D + \bar\Delta) \textit{ log-canonical pair,}\\ K_{\bar X} + \bar D + \bar\Delta \textit{ is ample,}\\ \tau\colon \bar D^\nu \to \bar D^\nu \textit{ is an involution,}\\ \mathrm{Diff}_{\bar D^\nu}(\Delta) \textit{ is } \tau\textit{-invariant.}\end{array}\right\}$$

We comment briefly on the proof. We have seen above that any stable log surface gives rise to a triple as above, where only the τ-invariance of the different remains

[4] See for example [Hac04], [Ale02], [HKT09], [Laz12].

[5] A local model for the pinch point in $\mathbb{A}^3$ is given by the equation $x^2 + yz^2$.

to be proved. This is not difficult, because $\omega_{\bar X}(\bar D)|_{\bar D^\nu} = \pi^*(\omega_X|_{D^\nu})$, see [Kol12, Prop. 5.12].

The other direction is more subtle. If we are given a triple $(\bar X, \bar D+\bar\Delta, \tau)$ we need to construct X and then prove that it is a stable log surface. First one uses results about quotients with respect to finite equivalence relations [Kol12, Cor. 5.30, Cor. 5.34] to construct X as an algebraic space. Then one has to prove that $K_X+\Delta$ is $\mathbb{Q}$-Cartier [Kol12, Prop. 5.15, Thm. 5.40] and finally conclude that $K_X+\Delta$ is ample on X by the version of Nakai-Moishezon valid for algebraic spaces [Kol90, Thm. 3.11]. Proving that the equivalence relation has the required properties and that the log-canonical bundle is $\mathbb{Q}$-Cartier becomes quite technical in higher dimensions.

Since we are especially interested in pluri-log-canonical maps and thus sections of pluri-log-canonical bundles, the following will play a role.

Proposition 2.14 ([Kol12, Prop. 5.8]) — *Let (X,Δ) be a stable log surface, $m\geq 1$. A section $s\in H^0(\bar X, \omega_{\bar X}(\bar D+\bar\Delta)^{[m]})$ descends to a section in $H^0(X, \omega_X(\Delta)^{[m]})$ if and only if its residue at the generic points of $\bar D^\nu$ is τ-invariant if m is even respectively τ-anti-invariant if m is odd.*

If $\omega_{\bar X}(\bar D+\bar\Delta)^{[m]}$ is a line bundle then this is equivalent to the image of s in $H^0(\bar D^\nu, \omega_{\bar D^\nu}^{[m]}(m\mathrm{Diff}_{\bar D^\nu}(\Delta)))$ being τ-invariant if m is even respectively τ-anti-invariant if m is odd.

The alternating signs are related to the Poincaré residue map: localising at a codimension 1 nodal point we look at the local model $\mathbb{A}^2 \supset X = Z(xy) = L_x\cup L_y$ so that a local generator for $\omega_{\mathbb{A}^2}(X)$ is $dx\wedge dy/xy$. Taking residues along the two lines we have

$$\mathrm{Res}_{L_x}\left(\frac{dx\wedge dy}{xy}\right) = \frac{dy}{y}, \quad \mathrm{Res}_{L_y}\left(\frac{dx\wedge dy}{xy}\right) = -\frac{dx}{x},$$

so they differ in sign at the node.

For later reference we also state

Proposition 2.15 — *Let X be a stable surface with normalisation $\bar X$. In the notation above we have $K_X^2 = (K_{\bar X}+\bar D)^2$ and $\chi(\mathcal{O}_X) = \chi(\mathcal{O}_{\bar X}) + \chi(\mathcal{O}_D) - \chi(\mathcal{O}_{\bar D})$.*

Proof. The first part is clear. For the second note that $\pi_*\mathcal{O}_{\bar X}(-\bar D)$ coincides with the ideal sheaf of the non-normal locus in X and thus we get the result by using additivity of the Euler characteristic for the two sequences

$$0\to \mathcal{O}_{\bar X}(-\bar D)\to\mathcal{O}_{\bar X}\to\mathcal{O}_{\bar D}\to 0,$$
$$0\to \pi_*\mathcal{O}_{\bar X}(-\bar D)\to\mathcal{O}_{X}\to\mathcal{O}_{D}\to 0.$$

□

2.3. **The curve embedding theorem.** The technique of restriction to curves will play a major role in our approach and thus we will often need the following numerical criterion due to Catanese, Franciosi, Hulek and Reid. We state it in a slightly weaker version that suffices for our purpose.

Theorem 2.16 ([CFHR99, Thm. 1.1]) — *Let C be a projective curve (over an algebraically closed field) which is Cohen–Macaulay but not necessarily irreducible or reduced and $\mathcal{L}$ a line bundle on C. Then $\mathcal{L}$ is base-point-free if for every generically Gorenstein subcurve $B\subset C$*

$$\deg_B\mathcal{L} = \chi(\mathcal{O}_B) - \chi(\mathcal{L}|_B) \geq 2p_a(B) = 2(1-\chi(\mathcal{O}_B))$$

and $\mathcal{L}$ is very ample if the inequality is strict.

Note that for an irreducible and smooth curve this gives the classical bounds.

3. A vanishing theorem on slc surfaces

We will need the following basic vanishing result, which is a variant of [KSS10, Cor. 6.6], and some consequences. All of these results follow from general vanishing theorems in [Fuj12] but the surface case is technically much simpler.

Proposition 3.1 (Generalised Kodaira vanishing) — *Let X be an slc surface and A a well-behaved, integral, ample, $\mathbb{Q}$-Cartier divisor on X. Then*

$$H^i(X, \mathcal{O}_X(-mA)) = 0 \qquad \text{for all } i < 2 \text{ and } m \geq 1.$$

Proof. Choose $k \in \mathbb{N}$ such that $\mathcal{O}_X(kA)$ is a very ample line bundle and let B be the divisor of a general section. In particular, B is a reduced divisor contained in the locus where X has at most normal crossing singularities and B is non-singular where X is smooth. Now consider the ramified simple cyclic cover

$$\pi\colon Y = \mathrm{Spec}_X\left(\bigoplus_{i=0}^{k-1} \mathcal{O}_X(-kA)\right) \to X$$

with the following properties:

- (i) Y is Cohen–Macaulay by construction since each $\mathcal{O}_X(-kA)$ is S_2 and Y is a surface.
- (ii) Y has Du Bois singularities by [Kol12, Cor. 6.20].
- (iii) $(\pi^*\mathcal{O}_X(A))^{[1]}$ is locally free [Kol12, 2.56.6] and ample because the pullback of an ample divisor via a finite map is ample.
- (iv) Let $Z \subset X$ be a codimension 2 subset such that $X \setminus Z$ is Gorenstein. For every reflexive sheaf $\mathcal{F}$ on X we have

  $$\pi_*\left(\pi^*\mathcal{F}\right)^{[1]} \cong \left(\pi_*\mathcal{O}_Y \otimes_{\mathcal{O}_X} \mathcal{F}\right)^{[1]} \cong \bigoplus_{i=0}^{k-1} \left(\mathcal{O}_X(-kA)\right) \otimes_{\mathcal{O}_X} \mathcal{F})^{[1]}$$

  because all sheaves above are S_2 and isomorphic over $X \setminus Z$ where π is flat.

Now by (iii) the sheaf $\pi^*(\mathcal{O}_X(-mA))^{[1]}$ is line bundle on Y whose inverse is ample for $m \geq 1$. Since Y is Cohen–Macaulay and Du Bois, the Du Bois version of Kodaira vanishing [Kol12, Thm. 9.41] implies for $i < 2$ and $m \geq 1$

$$\begin{aligned}
0 &= H^i(Y, (\pi^*\mathcal{O}_X(-mA))^{[1]}) \\
&= H^i(X, \pi_*(\pi^*\mathcal{O}_X(-mA))^{[1]}) \\
&= H^i\left(X, \bigoplus_{i=0}^{k-1} (\mathcal{O}_X(-kA)) \otimes_{\mathcal{O}_X} \mathcal{O}_X(-mA))^{[1]}\right) \qquad \text{by } (iv) \\
&= H^i\left(X, \bigoplus_{i=0}^{k-1} \mathcal{O}_X((-m-k)A)\right) \\
&\supset H^i\left(X, \mathcal{O}_X(-mA)\right)
\end{aligned}$$

This concludes the proof. □

Remark 3.2 — The reason for the restriction to dimension 2 in the above theorem is that the index-1-cover constructed in the proof may well fail to be Cohen–Macaulay in higher dimensions.

Lemma 3.3 — *Let X be a demi-normal surface, Δ a well-behaved, integral Weil divisor on X, and $\mathcal{F}, \mathcal{G}$ reflexive sheaves on X that are locally free outside codimension two. Then*

$$\mathrm{Ext}^1_X(\mathcal{F}(\Delta), \mathcal{G}) \cong \mathrm{Ext}^1_X(\mathcal{F}, \mathcal{G}(-\Delta)).$$

Proof. We first claim that $\mathscr{E}xt^1_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}) = 0$. Indeed, let $j\colon U \hookrightarrow X$ be the inclusion of an open subset where all sheaves in question are locally free and such that the complement is of codimension 2. Then by [Har66, Ch. 2, Prop. 5.8]

$$j^*\mathscr{E}xt^1_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}) = \mathscr{E}xt^1_{\mathcal{O}_U}(j^*\mathcal{F}, j^*\mathcal{G}) = 0$$

so that $\mathscr{E}xt^1_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ is torsion supported in codimension 2.

To study $\mathscr{E}xt^1_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ we may assume that X is affine. Since both $\mathcal{F}$ and $\mathcal{G}$ are reflexive, in any extension $0 \to \mathcal{G} \to \mathcal{E} \to \mathcal{F} \to 0$ the sheaf $\mathcal{E}$ is also reflexive. Thus the extension is determined outside codimension 2 as can be seen from the diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{G} \arrow[r] \arrow[d, equal] & \mathcal{E} \arrow[r] \arrow[d] & \mathcal{F} \arrow[r] \arrow[d, equal] & 0 . \\
0 \arrow[r] & j_*(j^*\mathcal{G}) \arrow[r] & j_*(j^*\mathcal{E}) \arrow[r] & j_*(j^*\mathcal{F}) \arrow[r] & 0
\end{tikzcd}$$

Thus the extension splits if it splits outside codimension two and $\mathscr{E}xt^1_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ has no torsion supported in codimension 2. By the above it is actually zero.

Returning to the claim of the lemma, the local-to-global-Ext-spectral-sequence induces an exact sequence

$$0 \to H^1(X, \mathscr{H}om_{\mathcal{O}_X}(\mathcal{F}(\Delta), \mathcal{G})) \to \mathrm{Ext}^1_X(\mathcal{F}(\Delta), \mathcal{G}) \to H^0(X, \mathscr{E}xt^1_{\mathcal{O}_X}(\mathcal{F}(\Delta), \mathcal{G}))$$

so by the above we have $\mathrm{Ext}^1_X(\mathcal{F}(\Delta), \mathcal{G}) \cong H^1(X, \mathscr{H}om_{\mathcal{O}_X}(\mathcal{F}(\Delta), \mathcal{G}))$. Since the sheaves $\mathscr{H}om_{\mathcal{O}_X}(\mathcal{F}(\Delta), \mathcal{G})$ and $\mathscr{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}(-\Delta))$ are both reflexive and isomorphic on U they are isomorphic and hence

$$\begin{aligned}\mathrm{Ext}^1_X(\mathcal{F}(\Delta), \mathcal{G}) &\cong H^1(X, \mathscr{H}om_{\mathcal{O}_X}(\mathcal{F}(\Delta), \mathcal{G}))\\ &\cong H^1(X, \mathscr{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}(-\Delta))) \cong \mathrm{Ext}^1_X(\mathcal{F}, \mathcal{G}(-\Delta))\end{aligned}$$

which concludes the proof. $\square$

Corollary 3.4 — *Let (X, Δ) be a stable log surface with global index I. Then for all $i > 0$ and all integers $m \geq 2$,*

(*i*) $H^i(X, \omega_X^{[m]}((m-1)\Delta)) = 0$,
(*ii*) $H^i(X, \omega_X(\Delta)^{[mI]}) = 0$.

In particular, if X is a stable surface (with empty boundary) then $H^i(X, \omega_X^{[m]}) = 0$ for all $i > 0$ and all integers $m \geq 2$.

Proof. By Serre duality on the Cohen–Macaulay scheme X and Lemma 3.3, we have

$$\begin{aligned}H^i\left(X, \omega_X^{[m]}((m-1)\Delta)\right)^* &\cong \mathrm{Ext}^{2-i}_X(\omega_X^{[m]}((m-1)\Delta), \omega_X)\\ &\cong H^{2-i}(X, \omega_X(\Delta)^{[1-m]}),\end{aligned}$$

so by Proposition 3.1, $\omega_X^{[m]}((m-1)\Delta)$ has no higher cohomology for $m \geq 2$ which proves the first item as well as the case without boundary.

For the second item consider the exact sequence

$$0 \to \omega_X^{[mI]}((mI-1)\Delta) \to \omega_X(\Delta)^{[mI]} \to \omega_X(\Delta)^{[mI]}|_\Delta \to 0,$$

whose long exact cohomology sequence gives

$$H^1(X, \omega_X(\Delta)^{[mI]}) \cong H^1(\Delta, \omega_X(\Delta)^{[mI]}|_\Delta), \qquad H^2(X, \omega_X(\Delta)^{[mI]}) = 0. \tag{1}$$

Thus we need to prove that $H^1(\Delta, \omega_X(\Delta)^{[mI]}|_\Delta) = 0$ for $m \geq 2$.

We argue by contradiction, so assume $H^1(\Delta, \omega_X(\Delta)^{[mI]}|_\Delta) \neq 0$. By Lemma A.2 there is a subcurve $B \subset \Delta$ with a generically onto morphism $\lambda_B \colon \omega_X(\Delta)^{[mI]}|_B \to \omega_B$. On the other hand, since (X, Δ) is a stable log surface, $\omega_X(\Delta)^{[I]}|_B$ is an ample line bundle and thus has positive degree on B. Furthermore $\deg(\omega_X(\Delta)^{[I]}|_B) \geq \deg(\omega_B^{\otimes I})$ because the different is effective, see Lemma 4.7. Hence for $m \geq 2$ the degree of $\omega_X(\Delta)^{[mI]}|_B$ is strictly larger than the degree of $\omega_B^{\otimes I}$ and there cannot exist a λ_B as above — a contradiction. Thus $H^1(\Delta,\ \omega_X(\Delta)^{[mI]}|_\Delta) = 0$ as claimed and equation (1) concludes the proof. ☐

Corollary 3.5 — *Let (X, Δ) be a stable log surface with global index I and $\mathcal{I}$ the ideal sheaf of $D \cup \Delta \subset X$. In the notation of Section 2.2 we have*

(i) $H^0(X, \mathcal{I}\omega_X(\Delta)^{[mI]}) = H^0(\bar{X}, \omega_{\bar{X}}(\bar{D} + \bar{\Delta})^{[mI]}(-\bar{D} - \bar{\Delta}))$ *for* $m \geq 0$,

(ii) $H^i(X, \mathcal{I}\omega_X(\Delta)^{[mI]}) = 0$ *for all* $i > 0$ *and all integers* $m \geq 2$.

Proof. Let $\bar{\mathcal{I}}$ be the ideal sheaf of $\bar{D} \cup \bar{\Delta}$ in $\bar{X}$. Then we have $\pi_*\bar{\mathcal{I}} = \mathcal{I}$. Now

$$\omega_{\bar{X}}(\bar{D} + \bar{\Delta})^{[mI]}(-\bar{D} - \bar{\Delta}) = \bar{\mathcal{I}}\omega_X(\bar{D} + \bar{\Delta})^{[mI]} = \bar{\mathcal{I}}\pi^*\omega_X(\Delta)^{[mI]}$$

and it follows that

$$\pi_*\omega_{\bar{X}}(\bar{D} + \bar{\Delta})^{[mI]}(-\bar{D} - \bar{\Delta}) = \pi_*\bar{\mathcal{I}} \otimes \omega_X(\Delta)^{[mI]} = \mathcal{I}\omega_X(\Delta)^{[mI]}.$$

Since π is a finite morphism, we have by the Leray spectral sequence that

$$H^i(X, \mathcal{I}\omega_X(\Delta)^{[mI]}) = H^i(\bar{X}, \omega_{\bar{X}}(\bar{D} + \bar{\Delta})^{[mI]}(-\bar{D} - \bar{\Delta})) \text{ for } i \geq 0.$$

In particular (i) is proved. By applying Corollary 3.4(i) to the pair $(\bar{X}, \bar{D} + \bar{\Delta})$, the second item also follows. ☐

The first item is also a direct consequence of Proposition 2.14.

4. Base-point-freeness of pluri-log-canonical maps

Many of the standard techniques do not work directly on non-normal and possibly reducible surfaces. Thus to prove base-point-freeness we first use a Reider-type result of Kawachi on the normalisation to produce pluri-log-canonical sections vanishing along the non-normal locus. Then we analyse the restriction of the pluri-log-canonical bundle to the non-normal locus directly. The overall result is

Theorem 4.1 — *Let (X, Δ) be a connected stable log surface of global index I.*

(i) *The line bundle $\omega_X(\Delta)^{[mI]}$ is base-point-free for $m \geq 4$.*

(ii) *The line bundle $\omega_X(\Delta)^{[mI]}$ is base-point-free for $m \geq 3$ if one of the following holds:*

  (a) $I \geq 2$.

(b) *There is no irreducible component* $\bar{X}_i$ *of the normalisation such that* $\left(\pi^*(K_X+\Delta)|_{\bar{X}_i}\right)^2 = 1$ *and the union of* Δ *and the non-normal locus is a nodal curve.*

(c) X *is normal and we do not have* $I = (K_X+\Delta)^2 = 1$.

This result is sharp in the sense that there are examples of smooth surfaces such that $\omega_X^{\otimes 3}$ has base-points [BHPV04, Remark on p.287]. However, the conditions in the theorem are by no means neccessary for base-point-freeness.

In the case of surfaces with canonical singularities the bi-canonical map is a morphism as soon as $K_X^2 \geq 5$. We will show in Example 7.2 that this does not generalise to stable surfaces, even irreducible ones.

Remark 4.2 — If $|mI(K_X+\Delta)|$ is base-point-free, the fact that K_X is ample implies that the pluri-log-canonical map $\varphi_{mI}\colon X \to \mathbb{P}^N$ does not contract any curve on X, hence defines a finite morphism from X to its reduced image.

4.1. **Base-point-freeness on the complement of boundary and non-normal locus.** We now analyse the base-point-freeness of pluri-log-canonical maps outside the non-normal locus and the boundary of a stable log surface, starting with the following auxillary result.

Proposition 4.3 — *Let* $(\bar{X}, \bar{\Delta})$ *be an irreducible (hence connected) log surface with log-canonical singularities,* $K_{\bar{X}} + \bar{\Delta}$ *ample and* I *the global index. Assume* $m \geq 3$ *if* $I \geq 2$ *and in addition* $(m-1)^2(K_{\bar{X}}+\bar{\Delta})^2 > 4$ *if the index is 1.*

Then for every $x \in \bar{X} \setminus \bar{\Delta}$ *there is a section of* $\omega_{\bar{X}}(\bar{\Delta})^{[mI]}(-\bar{\Delta})$ *not vanishing at* x*. In particular, the rational map associated to* $\omega_{\bar{X}}(\bar{\Delta})^{[mI]}$ *is a morphism on* $\bar{X} \setminus \bar{\Delta}$.

Proof. Let $\bar{f}\colon \bar{Y} \to \bar{X}$ be the minimal resolution of those singularities of $\bar{X}$ that are contained in $\bar{\Delta}$. We can write $K_{\bar{Y}} = \bar{f}^*(K_{\bar{X}}+\bar{\Delta}) - \Delta_{\bar{Y}} - E$, where $\Delta_{\bar{Y}}$ is the strict transform of $\bar{\Delta}$ and E is exceptional. Since $\bar{f}$ is the minimal resolution along $\bar{\Delta}$, the divisor E is effective.

Consider the divisor $\bar{M} = (mI-1)(K_{\bar{X}}+\bar{\Delta})$. The pullback $\bar{f}^*\bar{M}$ (in the sense of Mumford) is a big and nef $\mathbb{Q}$-divisor on $\bar{Y}$ and

$$\begin{aligned} K_{\bar{Y}} + \lceil \bar{f}^*\bar{M} \rceil &= \lceil K_{\bar{Y}} + \bar{f}^*\bar{M} \rceil \\ &= \lceil \bar{f}^*(K_{\bar{X}}+\bar{\Delta}) - \Delta_{\bar{Y}} - E + \bar{f}^*\bar{M} \rceil \\ &= \bar{f}^* mI(K_{\bar{X}}+\bar{\Delta}) - \Delta_{\bar{Y}} - \lfloor E \rfloor \end{aligned}$$

is Cartier. Thus, as soon as

$$\bar{f}^*\bar{M}^2 > 4 \text{ and } \bar{f}^*\bar{M}.C \geq 2 \tag{2}$$

holds for all curves C not contained in the exceptional locus of $\bar{f}$, we can apply [Kaw00, Thm. 2] to conclude that $K_{\bar{Y}} + \lceil \bar{f}^*\bar{M} \rceil$ is base-point-free outside the exceptional locus of $\bar{f}$. Since

$$\begin{aligned} H^0(\bar{Y}, \mathcal{O}_{\bar{Y}}(K_{\bar{Y}} + \lceil \bar{f}^*\bar{M} \rceil)) &= H^0(\bar{Y}, \mathcal{O}_{\bar{Y}}(f^*mI(K_{\bar{X}}+\bar{\Delta}) - \Delta_{\bar{Y}} - \lfloor E \rfloor)) \\ &\subset H^0(\bar{Y}, \mathcal{O}_{\bar{Y}}(f^*mI(K_{\bar{X}}+\bar{\Delta}) - \Delta_{\bar{Y}})) \\ &\cong H^0(\bar{X}, \bar{f}_*\mathcal{O}_{\bar{Y}}(f^*mI(K_{\bar{X}}+\bar{\Delta}) - \Delta_{\bar{Y}})) \\ &\cong H^0(\bar{X}, \mathcal{O}_{\bar{X}}(mI(K_{\bar{X}}+\bar{\Delta}) - \bar{\Delta})) \\ &= H^0(\bar{X}, \omega_{\bar{X}}(\bar{\Delta})^{[mI]}(-\bar{\Delta})) \end{aligned}$$

all sections of $K_{\bar{Y}} + \lceil \bar{f}^*\bar{M} \rceil$ descend to sections of $\omega_{\bar{X}}(\bar{\Delta})^{[mI]}(-\bar{\Delta})$. Via the inclusion $\omega_{\bar{X}}(\bar{\Delta})^{[mI]}(-\bar{\Delta}) \hookrightarrow \omega_{\bar{X}}(\bar{\Delta})^{[mI]}$ we can also interpret these as section of $\omega_{\bar{X}}(\bar{\Delta})^{[mI]}$

vanishing along $\bar\Delta$. The restriction $\bar f\colon \bar Y\setminus \bar f^{-1}\bar\Delta \to \bar X\setminus\bar\Delta$ is an isomorphism and thus base-point-freeness holds on $\bar X\setminus\bar\Delta$ under the assumption (2).

It remains to show that (2) holds in the cases given above. First we note that for $m\geq 3$ and for every non-exceptional curve $C\subset \bar Y$ we have

$$\bar f^*M.C=(mI-1)(K_{\bar X}+\bar\Delta)\bar f_*C=2I(K_{\bar X}+\bar\Delta)\bar f_*C+((m-2)I-1)(K_{\bar X}+\bar\Delta)\bar f_*C\geq 2$$

because $I(K_{\bar X}+\bar\Delta)$ is an ample Cartier divisor.

Noting that $I(K_{\bar X}+\bar\Delta)^2$ is a positive integer, because it is the intersection of an ample Cartier divisor with an integral divisor, and writing

$$\bar f^*\bar M^2=(Im-1)^2(K_{\bar X}+\bar\Delta)^2>4\Leftrightarrow \left(m-\frac{1}{I}\right)^2 I(K_{\bar X}+\bar\Delta)^2>\frac{4}{I}$$

we see that if $I\geq 2$ and $m\geq 2$ the inequality is satisfied. If $I=1$ we need that $(m-1)^2(K_{\bar X}+\bar\Delta)^2>4$ in addition to $m\geq 3$. $\square$

We now descend the above result to a possibly non-normal stable log surface.

Corollary 4.4 — *Let (X,Δ) be a stable log surface with global index I. Then the base-points of $\omega_X(\Delta)^{[mI]}$ are contained in the union of the non-normal locus D and the boundary Δ if*

(*i*) *$m\geq 4$,*

(*ii*) *$m\geq 3$ unless the index $I=1$ and there is an irreducible component $\bar X_i$ of the normalisation such that $\left(\pi^*(K_X+\Delta)|_{\bar X_i}\right)^2=1$.*

Proof. We use our standard notation from Section 2.2. On every irreducible component $\bar X_i$ of the normalisation $\bar X$, we apply Proposition 4.3 to the pair $(\bar X_i,(\bar\Delta+\bar D)|_{\bar X_i})$ which, under our assumptions, gives for every point $\bar x\in\bar X\setminus(\bar\Delta\cup\bar D)$ a section of $\omega_{\bar X}(\bar D+\bar\Delta)^{[mI]}(-\bar D-\bar\Delta)$ not vanishing at $\bar x$.

Via the inclusion

$$\omega_{\bar X}(\bar D+\bar\Delta)^{[mI]}(-\bar D-\bar\Delta)\hookrightarrow \omega_{\bar X}(\bar D+\bar\Delta)^{[mI]},$$

the sections of $\omega_{\bar X}(\bar D+\bar\Delta)^{[mI]}(-\bar D-\bar\Delta)$ are mapped to sections of $\omega_{\bar X}(\bar D+\bar\Delta)^{[mI]}$ that vanish along $\bar D\cup\bar\Delta$ and thus descend to sections of $\omega_X(\Delta)^{[mI]}$ by Proposition 2.14. Consequently, $\omega_X(\Delta)^{[mI]}$ has no base-points outside $\Delta\cup D$. $\square$

4.2. **Restricting to non-normal locus and boundary.** In this section we concentrate on the geometry of the non-normal locus D of a stable log surface (X,Δ), using the same notation as in Section 2.2. We start with a definition that will turn out to describe all possible singularities of the non-normal locus.

Definition 4.5 — Let B be a reduced curve, $p\in B$ and $\mu=\mu_p(B)$ the multiplicity of p in B. We call p a μ-multi-node, if locally analytically at p the curve is isomorphic to the union of the coordinate axes in $\mathbb{A}^\mu$.

Thus a 1-multi-nodal point is smooth and a 2-multi-node is just an ordinary nodal point.

By the correspondence between stable log surfaces and their normalisation explained after Theorem 2.13 (see also [Kol12, Thm. 8.12 (3)]), the non-normal locus D is quotient by the finite equivalence relation on $\bar D$ induced by τ. In other words, as

a set D is the set of equivalence classes of the equivalence relation on $\bar D^\nu$ generated by

$$\bar p \sim \bar q \text{ if } \tau(\bar p) = \bar q \text{ or } \nu(\bar p) = \nu(\bar q), \tag{3}$$

and the scheme structure on D is determined by the requirement that the diagram

$$\begin{tikzcd} \bar D \arrow[d, "\pi"] & \bar D^\nu \arrow[l, "\bar\nu"'] \arrow[d, "/\tau"] \\ D & D^\nu \arrow[l, "\nu"] \end{tikzcd}$$

is a pushout diagram.

Recall that $\bar D$ is a nodal curve by the classification of log-canonical singularities. If $\bar D$ is smooth then $\bar\nu$ is an isomorphism, so $D^\nu = \bar D/\tau$ satisfies the pushout property and $D = D^\nu$. Applying the same argument locally, it follows that if $p \in D$ is a point such that $\pi^{-1}(p)$ contains only smooth points of $\bar D$ then p itself is smooth.

Now let p be a singular point of D. Write the preimage of p as

$$(\pi \circ \bar\nu)^{-1}(p) = \{a_1, b_1, \dots, a_k, b_k, c_1, \dots, c_l\}$$

such that $\bar\nu(a_i) = \bar\nu(b_i)$ is a node of $\bar D$ and $\bar\nu(c_j)$ is a smooth point of $\bar D$. By the above there is at least one node mapping to p, so $k \geq 1$. Since the preimage is an equivalence class with respect to the relation generated by (3), we have $l \in \{0, 1, 2\}$ and, up to reordering, the following cases can occur (compare also [Kol10, 17.4]):

Type I: The preimage consists only of nodes and we glue in a circular fashion: $\tau(b_i) = a_{i+1}\,(i = 1, \dots, k-1)$, $\tau(b_k) = a_1$. In this case we get a smooth point of D if $k = 1$, so we assume $k \geq 2$.

Type II: We glue the preimages of the nodes in a chain $\tau(b_i) = a_{i+1}\,(i = 1, \dots, k-1)$ and have two remaining points a_1 and b_k at the ends. At each end we have two possibilities which we spell out only for a_1: either $\tau(a_1) = a_1$ or $\tau(a_1) = c_j$ for a point c_j mapping to a smooth point of $\bar D$.

To determine the local structure of D at p, we replace D by a small analytic neighbourhood of p such that p is the only singular point and τ has no fixed points on $\bar D^\nu \setminus (\pi \circ \bar\nu)^{-1}(p)$. Then, possibly shrinking D further, the normalisation $D^\nu = \bar D^\nu/\tau$ consists of k (Type I) or $k+1$ (Type II) branches, each containing a unique point that maps to p, and $\bar D$ consists of k nodal and possibly one or two smooth branches for Type II.

Thus there are maps from $\bar D$ and D^ν to a neighbourhood of a multi-nodal point compatible with the equivalence relation. These satisfy the pushout conditions because if the tangent directions were not independent then the map to D would factor over the multi-nodal point. So every singular point $p \in D$ is a μ-multi-nodal point for some $\mu \geq 2$.

By Theorem 2.13 the different $\mathrm{Diff}_{\bar D^\nu}(\bar\Delta)$ is τ-invariant and thus has the same coefficient for each point in an equivalence class of the relation generated by (3). In particular, if $p \in D$ is singular then the preimage contains at least one node and the different has coefficient 1 at each point mapping to p. This restricts the possibilities for the smooth points $\bar\nu(c_i) \in \bar D$ occuring in Type II: by the classification of log-canonical singularities they are either dihedral points of $\bar X$ or smooth points of $\bar D$ where $\bar D$ intersects $\bar\Delta$.

The first two items of the next lemma sum up the discussion so far.

Lemma 4.6 — *Let (X, Δ) be a log surface with slc singularities and $D \subset X$ the non-normal locus. Let $p \in D \cup \Delta$. Then the following holds.*

(*i*) *p is a μ-multi-node of $D \cup \Delta$ for $\mu = \mu_p(D \cup \Delta) \geq 1$.*
(*ii*) *If p is a singularity of $D \cup \Delta$ then the inverse image $\pi^{-1}(p)$ contains at least one node of $\bar{D} \cup \bar{\Delta}$ and at each point of $\bar{D}^\nu$ mapping to p the different $\mathrm{Diff}_{\bar{D}^\nu}(\bar{\Delta})$ has coefficient 1.*
(*iii*) *Let $B \subset D \cup \Delta$ be a subcurve. Then $p_a(B) = p_a(B^\nu) + \sum_p(\mu_p(B) - 1)$.*

Proof. It remains to prove the last item. We compare the arithmetic genera of B and B^ν by taking Euler characteristics in the short exact sequence

$$0 \to \mathcal{O}_B \to \nu_*\mathcal{O}_{B^\nu} \to \nu_*\mathcal{O}_{B^\nu}/\mathcal{O}_B \to 0.$$

Since all singular points are μ-multi-nodes the length of the subsheaf of $\nu_*\mathcal{O}_{B^\nu}/\mathcal{O}_B$ supported at the point p is exactly $\mu_p(B) - 1$, so $p_a(B) = p_a(B^\nu) + \sum_p(\mu_p(B) - 1)$. □

Now we analyse the restriction of the log-canonical bundle to the non-normal locus and the boundary.

Lemma 4.7 — *Let (X, Δ) be a stable log surface with normalisation $(\bar{X}, \bar{D} + \bar{\Delta})$. Consider a subcurve $B \subset D \cup \Delta$, with normalisation B^ν. Let s be the number of smooth points of B that are singular points of $D \cup \Delta$. Then*

$$\begin{aligned}\deg(K_X + \Delta)|_{B^\nu} &\geq 2p_a(B^\nu) - 2 + \sum_{p \in (D\cup\Delta)_{\text{sing}}} \mu_p(B)\\ &\geq 2p_a(B) - 2 - \sum_{p \in B_{\text{sing}}} (2 - \mu_p(B)) + s\\ &\geq 2p_a(B) - 2 - \sum_{p \in B_{\text{sing}}} (2 - \mu_p(B))\end{aligned}$$

Proof. Let $B_1 = B \cap D$ and $B_2 = B \cap \Delta$, that is, B_1 (resp. B_2) is the subcurve of B contained in D (resp. Δ). As Weil divisors we can write $D + \Delta = A + B = A_i + B_i$ with A (resp. A_i) being the complement curve of B (resp. B_i). We adopt our usual notation for strict transforms in $\bar{X}$ and normalisation: for example, $\bar{B}$ is the strict transform of B in $\bar{X}$, B^ν is the normalisation of B while $\bar{B}^\nu$ is the normalisation of $\bar{B}$.

Note that $\pi_1 \colon \bar{B}_1^\nu \to B_1^\nu$ is a double cover and $\pi_2 \colon \bar{B}_2^\nu \to B_2^\nu$ is an isomorphism. Thus by Hurwitz formula

$$K_{\bar{B}_1^\nu} = \pi^* K_{B_1^\nu} + R \text{ and } K_{\bar{B}_2^\nu} = \pi^* K_{B_2^\nu} \tag{4}$$

where R is the (reduced) ramification divisor on $\bar{B}_1^\nu$.

Now we compute the degree of $K_X + \Delta$ restricted to B:

$$
\begin{aligned}
\deg(K_X+\Delta)|_B &= \deg(K_X+\Delta)|_{B_1} + \deg(K_X+\Delta)|_{B_2}\\
&= \frac{1}{2}\deg(K_{\bar X}+\bar D+\bar\Delta)|_{\bar B_1^\nu} + \deg(K_{\bar X}+\bar D+\bar\Delta)|_{\bar B_2^\nu}\\
&= \frac{1}{2}\deg(K_{\bar B_1^\nu}+\mathrm{Diff}_{\bar B_1^\nu}(\bar A_1)) + \deg(K_{\bar B_2^\nu}+\mathrm{Diff}_{\bar B_2^\nu}(\bar A_2))\\
&\overset{(4)}{=} \deg\left(\frac{1}{2}\left(\pi_1^*K_{B_1^\nu}+R+\mathrm{Diff}_{\bar B_1^\nu}(\bar A_1)\right)+\pi_2^*K_{B_2^\nu}+\mathrm{Diff}_{\bar B_2^\nu}(\bar A_2)\right) \qquad (5)\\
&= \deg(K_{B_1^\nu}+K_{B_2^\nu}) + \deg\left(\frac{1}{2}\left(\mathrm{Diff}_{\bar B_1^\nu}(\bar A_1)+R\right)+\mathrm{Diff}_{\bar B_2^\nu}(\bar A_2)\right)\\
&= \deg K_{B^\nu} + \deg\left(\frac{1}{2}\left(\mathrm{Diff}_{\bar B_1^\nu}(\bar A_1)+R\right)+\mathrm{Diff}_{\bar B_2^\nu}(\bar A_2)\right)
\end{aligned}
$$

Let p be a point in B such that $\mu_p(D\cup\Delta)\geq 2$. Then we can decompose the multiplicity $\mu_p(B)=\mu_p(B_1)+\mu_p(B_2)$. Taking the degree of the maps into account we have

$$
\begin{aligned}
\mu_p(B_1) &= \frac{1}{2}\#(\nu_1\circ\pi_1)^{-1}(p) + \frac{1}{2}\#\left((\nu_1\circ\pi_1)^{-1}(p)\cap R\right),\\
\mu_p(B_2) &= \#(\nu_2\circ\pi_2)^{-1}(p),
\end{aligned} \qquad (6)
$$

where $\nu_i\colon B_i^\nu\to B_i$, $i=1,2$, are the normalisations and R is the set of ramification points of π_1.

The different $\mathrm{Diff}_{\bar B^\nu}(\bar A)=\mathrm{Diff}_{\bar B_1^\nu}(\bar A_1)+\mathrm{Diff}_{\bar B_2^\nu}(\bar A_2)$ is effective and has coefficient 1 over every singular point of $D\cup\Delta$ which lies in B (Lemma 4.6). Combining this with (5) and (6), we have

$$
\begin{aligned}
\deg(K_X+\Delta)|_B &\geq \deg K_{B^\nu} + \sum_{p\in(D\cup\Delta)_{\mathrm{sing}}}(\mu_p(B_1)+\mu_p(B_2))\\
&= 2p_a(B^\nu)-2+\sum_{p\in(D\cup\Delta)_{\mathrm{sing}}}\mu_p(B)
\end{aligned}
$$

and using Lemma 4.6(*iii*)

$$
\begin{aligned}
&\geq 2p_a(B)-2+s+\sum_{p\in B_{\mathrm{sing}}}\mu_p(B)-2(\mu_p(B)-1)\\
&= 2p_a(B)-2+s+\sum_{p\in B_{\mathrm{sing}}}2-\mu_p(B)\\
&\geq 2p_a(B)-2+\sum_{p\in B_{\mathrm{sing}}}2-\mu_p(B).
\end{aligned}
$$

This concludes the proof. □

Proposition 4.8 — *Let (X,Δ) be a stable log surface with global index I.*

(*i*) *If $m\geq 4$ then the line bundle $\omega_X(\Delta)^{[mI]}|_{D\cup\Delta}$ is base-point-free and the associated morphism is birational.*

(*ii*) *If $m\geq 3$ and $I\geq 2$ then the line bundle $\omega_X(\Delta)^{[mI]}|_{D\cup\Delta}$ is base-point-free.*

(*iii*) *If $m\geq 2$ and $D\cup\Delta$ is a nodal curve then the line bundle $\omega_X(\Delta)^{[mI]}|_{D\cup\Delta}$ is base-point-free.*

In each case the line bundle is very ample if the inequality is strict.

Proof. By the curve embedding theorem 2.16 it suffices to check that for every (Cohen–Macaulay) subcurve $B \subset D \cup \Delta$ we have $\deg mIK_X|_B \geq 2p_a(B)$ for base-point-freeness respectively $\deg mIK_X|_B \geq 2p_a(B)+1$ for very ampleness. We concentrate on the base-point-freeness now, the proof for very ampleness being the same.

We start with (iii) where B is a nodal curve. Recall that $I(K_X+\Delta)$ is an ample Cartier divisor and thus has degree at least one on each irreducible component of B. If $p_a(B) \leq 1$ the inequality $\deg mIK_X|_B \geq 2p_a(B)$ is trivially satisfied for $m \geq 2$. If $p_a(B) \geq 2$ then by Lemma 4.7

$$\deg I(K_X+\Delta)|_B \geq \deg(K_X+\Delta)|_B \geq 2p_a(B)-2+\sum_{p\in B_{\mathrm{sing}}} 2-\mu_p(B) = 2p_a(B)-2 \geq 2$$

and thus $\deg 2I(K_X+\Delta)|_B \geq 2p_a(B)$.

Now suppose B is singular. We compute, using Lemma 4.7,

$$\begin{aligned}
\deg 2(K_X+\Delta)|_{B^\nu} &= \deg(K_X+\Delta)|_{B^\nu} + \deg(K_X+\Delta)|_{B^\nu}\\
&\geq 2p_a(B^\nu)-2+\sum_{p\in B_{\mathrm{sing}}}\mu_p(B)+2p_a(B)-2-\sum_{p\in B_{\mathrm{sing}}}(2-\mu_p(B))\\
&= 2p_a(B^\nu)-4+2\sum_{p\in B_{\mathrm{sing}}}(\mu_p(B)-1)\\
&\geq 2p_a(B)+2p_a(B^\nu)-4+2\#\{p\in B \mid \mu_p(B)\geq 2\}\\
&\geq 2p_a(B)+2(p_a(B^\nu)-1) \qquad \text{(because } B \text{ is singular)}
\end{aligned}$$

If B has a irreducible components then $p_a(B^\nu)-1 = -\chi(\mathcal{O}_{B^\nu})$ is at least $-a$ and thus we continue the computation to get

$$\deg 2(K_X+\Delta)|_{B^\nu} \geq 2p_a(B)-2a \geq 2p_a(B)-2I\deg IK_X|_{B^\nu},$$

where the second inequality follows because the degree of the line bundle IK_X is at least 1 on each irreducible component of B^ν.

Therefore we have $\deg(2I+2)K_X|_{B^\nu} \geq 2p_a(B)$. Consequently if $m \geq \frac{2+2I}{I}$ then $\deg mIK_X|_B \geq 2p_a(B)$. This proves the base-point-freeness in case (i) and (ii).

It remains to prove that $\omega_X(\Delta)^{[mI]}|_{D\cup\Delta}$ gives a birational map for $m \geq 4$. Since this map is very ample if $I \geq 2$ or $m > 4$ by the above, we may assume that $I = 1$ and $m = 4$.

Let p, q be two smooth points of $D\cup\Delta$ and let $\mathcal{I} = \mathcal{O}_{D\cup\Delta}(-p-q) \subset \mathcal{O}_{D\cup\Delta}$ be the corresponding (invertible) ideal sheaf. To show that $\omega_X(\Delta)^{[4]}$ separates p and q is suffices to prove $H^1(D\cup\Delta, \omega_X(\Delta)^{[4]}\otimes\mathcal{I}) = 0$. Assume on the contrary $H^1(D\cup\Delta, \omega_X(\Delta)^{[4]}\otimes\mathcal{I}) \neq 0$. Then, by Lemma A.2, there is non-empty subcurve $B \subset D\cup\Delta$ such that

$$\chi(B, \omega_X(\Delta)^{[4]}\otimes\mathcal{I}|_B) \leq \chi(B, \omega_B) \tag{7}$$

with equality if and only if $\omega_X(\Delta)^{[4]}\otimes\mathcal{I}|_B \cong \omega_B$. On the other hand, we have shown above that $\deg \omega_X(\Delta)^{[4]}\otimes\mathcal{I}|_B \geq 2p_a(B)-2$ with strict inequality if B is nodal. Hence B is not a nodal curve, (7) is in fact an equality and $\omega_B \cong \omega_X(\Delta)^{[4]}\otimes\mathcal{I}|_B$ is a line bundle. But the dualising sheaf a curve with a μ-multi-node with $\mu \geq 3$ is not a line bundle by [Kol12, Aside 5.9] — a contradiction. □

Remark 4.9 — Examining the proof closely the bounds can be improved under additional assumptions, for example if there are no rational irreducible components of $D \cup \Delta$ on which ω_X has small degree.

Numerically, the worst case occurs if X is Gorenstein, the non-normal locus D has just one singular point, and every irreducible component B of D is rational with a 3-multi-nodal point such that $\deg(\omega_X|_B) = 1$. In this case, denoting by k the number of irreducible components of D, we have $p_a(D) = 2k$ and hence $\deg(\omega_X^{\otimes 3}|_D) = 2p_a(D) - k$. Such examples can be constructed explicitly, see [LR12a]. We will analyse the simplest such curve, a rational curve with a single 3-multi-node in Example 7.3. We will see that the conditions of Proposition 4.8 are not neccessary for base-point-freeness on such a curve but the bound for very ampleness is sharp: $\omega_X(\Delta)^{[4I]}|_{D\cup\Delta}$ need not be an embedding.

The following technical result will be used later on.

Corollary 4.10 — *Let (X, Δ) be a stable log surface with global index I. Assume $m \geq 3$ if $D \cup \Delta$ is a nodal curve and $m \geq 4$ otherwise.*

Then for each irreducible component $B \subset D \cup \Delta$ the image of the restriction map

$$\rho_B \colon H^0(D, \omega_X(\Delta)^{[mI]}|_{D\cup\Delta}) \to H^0(B, \omega_X(\Delta)^{[mI]}|_B)$$

has dimension at least three.

We will see in Example 7.3 that the bound cannot be improved if $D \cup \Delta$ is not a nodal curve: it is possible that $h^0(B, \omega_X(\Delta)^{[mI]}|_B) = 2$ for $m = 3$.

Proof. Let B be an irreducible component of D and n the dimension of the image of the restriction map ρ_B. By Proposition 4.8 the map induced by $\omega_X(\Delta)^{[mI]}|_{D\cup\Delta}$ is a birational morphism if $m \geq 4$ and an embedding if $m = 3$ and $D \cup \Delta$ is nodal. Thus $\operatorname{im} \rho_B$ defines a birational morphism $\varphi \colon B \to \varphi(B) \subset \mathbb{P}^{n-1}$ and $\varphi(B)$ is a non-degenerate curve of degree $\deg \omega_X(\Delta)^{[mI]}|_B \geq 3$. Therefore $\dim_{\mathbb{C}} \operatorname{im} \rho_B \geq 3$ as claimed. $\square$

4.3. **Proof of Theorem 4.1.** Let $m \geq 4$, or $m \geq 3$ if one of the conditions in Theorem 4.1(ii) holds. Note that $(ii)(b)$ implies $(ii)(c)$ by the classification of log-canonical singularities with reduced boundary.

The restriction map $\gamma \colon H^0(X, \omega_X(\Delta)^{[mI]}) \to H^0(D \cup \Delta, \omega_X(\Delta)^{[mI]}|_{D\cup\Delta})$ is surjective by the vanishing in Corollary 3.5. By Proposition 4.8 and the surjectivity of γ, we know that $\omega_X(\Delta)^{[mI]}$ has no base-points on $D \cup \Delta$ under our assumptions.

Combining this with Corollary 4.4 we conclude that $\omega_X(\Delta)^{[mI]}$ has no base-points for $m \geq 4$ and for $m \geq 3$ under the conditions given in Theorem 4.1(ii).

Remark 4.11 — According to the proof of Theorem 4.1, φ_{mI} $(m \geq 4)$ already separates the points on different irreducible components of X, that is, the image has the same number of irreducible components; in addition no normal point of X is mapped to the image of $D \cup \Delta$.

5. Pluri-log-canonical embeddings

In this section we prove our results on pluri-log-canonical maps. For a general stable log surface we can prove the following:

Theorem 5.1 — *Let (X, Δ) be a connected stable log surface of global index I.*

(i) *The line bundle $\omega_X(\Delta)^{[mI]}$ is very ample for $m \geq 8$.*

(*ii*) *The line bundle* $\omega_X(\Delta)^{[mI]}$ *defines a birational morphism for* $m \geq 6$.
(*iii*) *The line bundle* $\omega_X(\Delta)^{[mI]}$ *is very ample for* $m \geq 6$ *if* $I \geq 2$.

Under further assumptions on singularities and invariants we can improve these bounds.

Theorem 5.2 — *Let* (X, Δ) *be a connected stable log surface of global index* I.

(*i*) *The line bundle* $\omega_X(\Delta)^{[mI]}$ *is very ample for* $m \geq 7$ *if one of the following holds:*
  (a) *There is no irreducible component* $\bar{X}_i$ *of the normalisation such that* $\left(\pi^*(K_X + \Delta)|_{\bar{X}_i}\right)^2 = 1$ *and the union of* Δ *and the non-normal locus is a nodal curve.*
  (b) X *is normal and not* $(K_X + \Delta)^2 = 1$.

(*ii*) *The line bundle* $\omega_X(\Delta)^{[mI]}$ *is very ample for* $m \geq 6$ *if the normalisation* $\bar{X}$ *is smooth along the conductor divisor and has at most canonical singularities elsewhere.*

(*iii*) *The line bundle* $\omega_X(\Delta)^{[mI]}$ *is very ample for* $m \geq 5$ *if* $D \cup \Delta$ *is a nodal curve,* $\bar{X}$ *is smooth along the conductor divisor, and* $X \setminus D$ *has at most canonical singularities. In particular these conditions are satisfied, if* (X, Δ) *has semi-canonical singularities.*

We cannot prove and do not believe all bounds given in the Theorems to be sharp; we will discuss some evidence for this in Remark 5.8.

Remark 5.3 — One should resist the temptation to believe that X is Gorenstein if the normalisation is smooth: for example if we take $\bar{D} \subset \bar{X}$ to be the coordinate axes in the plane and pinch both lines via the restriction of $p \mapsto -p$ then the resulting slc surface has index 2. A local trivialising section of $\omega_{\bar{X}}(\bar{D})$ is $\frac{dx \wedge dy}{xy}$ and the residue along the x-axis $-dx/x$ is invariant under the involution and thus does not descend to X by Proposition 2.14.

However, in this case the index is always at most 2.

5.1. **Outline of the proof and preliminary results.** The strategy of our proof is classical: for every subscheme ξ of length two, find an appropriate pluri-log-canonical curve C containing it, and then prove that $mI(K_X+\Delta)$ embeds this curve and hence ξ by the numerical criterion of Theorem 2.16. There are two obstacles that restrict the choice of a curve C that we can handle:

- If C has a common irreducible component with $D \cup \Delta$ then it does not behave well under normalisation on which we need to compute intersection numbers and the adjunction.
- If $(\bar{X}, \bar{\Delta} + \bar{D})$ has worse than canonical singularites then some of the connectedness properties of ample curves are needed to counter-weigh the contributions from singularities, so in this case we can only effectively handle reduced curves C.

We start with some preliminary considerations how to construct well-behaved curves and how to get around the failure of the adjunction formula. At the end we include a version of the connectedness lemma for ample Cartier divisors on normal surfaces.

5.1.1. *Construction of well-behaved curves.*

Lemma 5.4 — *Let* (X, Δ) *be a stable log surface and* $\xi \subset X$ *a subscheme of length 2.*

(i) *If $\omega_X(\Delta)^{[mI]}$ has no base-points but $\varphi_{mI}|_\xi$ is not an embedding then there exists a well-behaved reduced curve $C \in |mI(K_X+\Delta)|$ containing ξ.*

(ii) *Assume $m \geq 3$ and $m \geq 4$ if $D \cup \Delta$ is not a nodal curve. Then there exists a well-behaved curve C in $|mI(K_X+\Delta)|$ containing ξ.*

In addition, if $|mI(K_X+\Delta)|$ has no base-points then C can be chosen to be reduced unless $\varphi_{mI}|_\xi$ is an embedding and the line spanned by $\varphi_{mI}(\xi)$ is contained in the branch locus of φ_{mI}.

Proof. Let $\xi \subset X$ be an arbitrary subscheme of length 2 and $\mathcal{I}$ its ideal sheaf. Recall that if φ_{mI} is a morphism then it is automatically finite (Remark 4.2).

In case (i) let $Z \subset \mathbb{P}^{h^0(\omega(\Delta)^{[mI]})-1}$ be the image of φ_{mI} and p be the image of ξ, a reduced point. Then the preimage of every hyperplane containing p contains ξ and the base locus of this linear system of hyperplanes is exactly the point p. Thus a general hyperplane section of Z containing p is reduced by Bertini and does not contain any irreducible component of the image of the non-normal locus or of the branch locus of φ_{mI}. Pullback gives the required curve C.

We now prove (ii). First assume that $D \cup \Delta$ is empty, i.e., we are on a normal surface without boundary. Then by Blache's version of Riemann–Roch for normal surfaces [Bla95, 3.4, 3.3(c), 2.1(d)] and Corollary 3.4, which applies as $m \geq 3$, we have

$$\begin{aligned} h^0(X, \omega_X^{[mI]}) &= \chi(\mathcal{O}_X(mIK_X) = \chi(\mathcal{O}_X) + \frac{m(mI-1)}{2} IK_X^2 \\ &\geq \chi(\mathcal{O}_X) + \frac{m(mI-1)}{2} \geq \chi(\mathcal{O}_X) + 3 \geq 4 \end{aligned}$$

where in the last step we used $\chi(\mathcal{O}_X) \geq 1$ from [Bla94, Theorem 2]. Thus there are at least 4 sections and at least a 2-dimensional space of sections vanishing on ξ.

Now assume $D \cup \Delta$ is non-empty. Consider, for an irreducible component B of $D \cup \Delta$, the diagram with exact rows and columns

$$\begin{tikzcd} 0 \arrow[d] & & \\ H^0(X, \omega_X(\Delta)^{[mI]} \otimes \mathcal{I}) \arrow[d] \arrow[dr, "\psi"] & & \\ H^0(X, \omega_X(\Delta)^{[mI]}) \arrow[r] & H^0(D, \omega_X(\Delta)^{[mI]}|_D) \arrow[r] \arrow[d, "\rho_B"] & 0 \\ & H^0(B, \omega_X(\Delta)^{[mI]}|_B). & \end{tikzcd}$$

By Corollary 4.10 the kernel of ρ_B has codimension at least three under our assumptions while the image of ψ has codimension at most two. Thus a general section in $H^0(X, \omega_X(\Delta)^{[mI]} \otimes \mathcal{I})$ does not restrict to zero on any irreducible component of $D \cup \Delta$.

Now assume in addition that $|mI(K_X+\Delta)|$ has no base-points. Because of (i), to get a reduced curve we only need to consider the case where $\varphi_{mI}|_\xi$ is an embedding. In that case $\varphi_{mI}(\xi)$ spans a line which is the base locus of the linear system of hyperplanes whose preimage contains ξ. Since a general curve $C \in |mI(K_X+\Delta)|$ that contains ξ is well-behaved, as is shown above, we can choose a hyperplane with

reduced preimage if and only if the line is not contained in the branch locus of φ_{mI}. □

5.1.2. *Corrections to adjunction.* Now our aim is to bound the correction terms occuring in the adjunction formula for a well-behaved curve on a stable log surface.

Let (X, Δ) be a stable log surface. We consider the minimal semi-resolution $f\colon Y \to X$. Let $\eta\colon \bar{Y} \to Y$ be the normalisation whose conductor divisor is denoted by $D_{\bar{Y}}$. The map $\bar{Y} \to X$ factors through $\bar{X}$ the normalisation of X, whose conductor divisor will now be denoted $D_{\bar{X}}$ (instead of $\bar{D}$); we get a commutative diagram

$$\begin{array}{ccc} \bar{Y} & \xrightarrow{\bar{f}} & \bar{X} \\ {\scriptstyle\eta}\downarrow & & {\scriptstyle\pi}\downarrow \\ Y & \xrightarrow{f} & X. \end{array}$$

Let $B \subset X$ be a well-behaved curve. We fix some notation to formulate the first result:

- $B_{\bar{Y}} \subset \bar{Y}$ and $B_{\bar{X}} \subset \bar{X}$ are the strict transforms of B.
- $\hat{B}_Y \subset Y$ is the hat transform of B, defined in Appendix A.
- $\hat{\Gamma}_{B_{\bar{X}}} = \hat{B}_{\bar{Y}} - B_{\bar{Y}}$ and $\Gamma^*_{B_{\bar{X}}} = \bar{f}^* B_{\bar{X}} - B_{\bar{Y}}$.
- $\Delta_{\bar{X}} \subset \bar{X}$ the strict transform of Δ.
- $\Lambda = \bar{f}^*(K_{\bar{X}} + D_{\bar{X}}) - (K_{\bar{Y}} + D_{\bar{Y}})$ is the codiscrepancy of the pair $(\bar{X}, D_{\bar{X}})$. Note that $\Lambda \geq 0$ because $\bar{f}\colon \bar{Y} \to \bar{X}$ is the normalisation of the minimal semi-resolution.

In the next lemma we estimate the failure of adjunction on the singular surface X in terms of data on the normalisation of the semi-resolution.

Lemma 5.5 — *Let (X, Δ) be a stable log surface and let $B \subset X$ be a well-behaved, not neccessarily reduced curve. Then with the above notation we have*

$$(K_X + \Delta + B)B \geq (K_X + B)B \geq 2p_a(B) - 2 + (\Lambda - \hat{\Gamma}_{B_{\bar{X}}} + \Gamma^*_{B_{\bar{X}}})(\hat{\Gamma}_{B_{\bar{X}}} - \Gamma^*_{B_{\bar{X}}}).$$

Proof. Since B is well-behaved, $\Delta B = \Delta_{\bar{X}} B_{\bar{X}} \geq 0$ and it suffices to prove the second inequality. By Proposition A.22, $p_a(B) \leq p_a(\hat{B}_{\bar{Y}}) + \frac{\hat{B}_{\bar{Y}} \hat{D}_{\bar{Y}}}{2}$, and hence, by adjunction on $\bar{Y}$, we have

$$2p_a(B) - 2 \leq (K_{\bar{Y}} + D_{\bar{Y}} + \hat{B}_{\bar{Y}})\hat{B}_{\bar{Y}}.$$

It follows that

$$\begin{aligned}
&(K_X + B)B - (2p_a(B) - 2) \\
&= (K_{\bar{X}} + D_{\bar{X}} + B_{\bar{X}})B_{\bar{X}} - (2p_a(B) - 2) \\
&\geq (K_{\bar{X}} + D_{\bar{X}} + B_{\bar{X}})B_{\bar{X}} - (K_{\bar{Y}} + D_{\bar{Y}} + \hat{B}_{\bar{Y}})\hat{B}_{\bar{Y}} \\
&= (K_{\bar{Y}} + D_{\bar{Y}} + \Lambda + B_{\bar{Y}} + \Gamma^*_{B_{\bar{X}}})B_{\bar{Y}} - (K_{\bar{Y}} + D_{\bar{Y}} + B_{\bar{Y}} + \hat{\Gamma}_{B_{\bar{X}}})(B_{\bar{Y}} + \hat{\Gamma}_{B_{\bar{X}}}) \\
&= (\Lambda + \Gamma^*_{B_{\bar{X}}} - \hat{\Gamma}_{B_{\bar{X}}})B_{\bar{Y}} - (K_{\bar{Y}} + D_{\bar{Y}} + B_{\bar{Y}} + \hat{\Gamma}_{B_{\bar{X}}})\hat{\Gamma}_{B_{\bar{X}}}
\end{aligned}$$

and since $B_{\bar{Y}}\bar{E}_i = -\Gamma^*_{B_{\bar{X}}}\bar{E}_i$ and $(K_{\bar{Y}} + D_{\bar{Y}} + \Lambda)\bar{E}_i = \pi^* K_X \bar{E}_i = 0$ for any i

$$\begin{aligned}
&= (\Lambda + \Gamma^*_{B_{\bar{X}}} - \hat{\Gamma}_{B_{\bar{X}}})(-\Gamma^*_{B_{\bar{X}}}) - (-\Lambda - \Gamma^*_{B_{\bar{X}}} + \hat{\Gamma}_{B_{\bar{X}}})\hat{\Gamma}_{B_{\bar{X}}} \\
&= (\Lambda - \hat{\Gamma}_{B_{\bar{X}}} + \Gamma^*_{B_{\bar{X}}})(\hat{\Gamma}_{B_{\bar{X}}} - \Gamma^*_{B_{\bar{X}}}),
\end{aligned}$$

Bringing $2p_a(B) - 2$ to the other side gives the second inequality. □

Lemma 5.6 — *Let (X, Δ) be a stable log surface of global index I, $m \geq 1$ and let $C \in |mI(K_X + \Delta)|$ be a well-behaved reduced curve. Then for every subcurve $B \subset C$ we have, in the notation introduced in Section 5.1.2,*

$$(mI + 1)(K_X + \Delta)B = (mI + 1)(K_{\bar{X}} + D_{\bar{X}} + \Delta_{\bar{X}})B_{\bar{X}} \geq 2p_a(B) - 2.$$

Proof. The first equality is clear and we only prove the second. We decompose $C = A + B$ as a Weil-divisor and let $A_{\bar{X}}$, $A_{\bar{Y}}$, $C_{\bar{X}}$, and $C_{\bar{Y}}$ be the strict transform of A and C in $\bar{X}$ resp. $\bar{Y}$.

We have

$$\begin{aligned}
mI(K_{\bar{X}} + D_{\bar{X}} + \Delta_{\bar{X}})B_{\bar{X}} - B_{\bar{X}}^2 &= (C_{\bar{X}} - B_{\bar{X}})B_{\bar{X}} \\
&= A_{\bar{X}}B_{\bar{X}} \\
&= (\bar{f}^*A_{\bar{X}})(\bar{f}^*B_{\bar{X}}) \\
&= (A_{\bar{Y}} + \Gamma^*_{A_{\bar{X}}})B_{\bar{Y}} \\
&\geq \Gamma^*_{A_{\bar{X}}}B_{\bar{Y}} \qquad \text{(since } C_{\bar{Y}} \text{ is reduced, } A_{\bar{Y}}B_{\bar{Y}} \geq 0) \\
&= -(\Gamma^*_{C_{\bar{X}}} - \Gamma^*_{B_{\bar{X}}})\Gamma^*_{B_{\bar{X}}}
\end{aligned}$$

where in the last line we use that $\bar{E}(B_{\bar{Y}} + \Gamma_{B_{\bar{X}}}) = 0$ for every $\bar{f}$-exceptional curve $\bar{E}$.

Adding this to the equation resulting from Lemma 5.5 we get

$$\begin{aligned}
(8)\quad & (mI + 1)(K_{\bar{X}} + D_{\bar{X}} + \Delta_{\bar{X}})B_{\bar{X}} \\
&\geq 2p_a(B) - 2 - (\hat{\Gamma}_{B_{\bar{X}}} - \Gamma^*_{B_{\bar{X}}})(\hat{\Gamma}_{B_{\bar{X}}} - \Lambda) - \Gamma^*_{B_{\bar{X}}}(\Gamma^*_{C_{\bar{X}}} - \hat{\Gamma}_{B_{\bar{X}}}).
\end{aligned}$$

By the definition of hat transform (Definition A.18) the intersection numbers of $\hat{\Gamma}_{B_{\bar{X}}} - \Gamma^*_{B_{\bar{X}}}$ and $\Gamma^*_{B_{\bar{X}}}$ with any exceptional divisor of $\bar{f}$ are non-positive. On the other hand $\Gamma^*_{C_{\bar{X}}} - \hat{\Gamma}_{B_{\bar{X}}} \geq 0$ by Lemma A.19(*iii*). Also $\hat{\Gamma}_{B_{\bar{X}}} - \Lambda$ has non-negative coefficients at every exceptional divisors mapped to $B_{\bar{X}}$ because at each of those $\hat{\Gamma}_{B_{\bar{X}}}$ has coefficients at least 1 while the coefficients of Λ are at most 1 for the log-canonical pair $(\bar{X}, D_{\bar{X}})$. So

$$-(\hat{\Gamma}_{B_{\bar{X}}} - \Gamma^*_{B_{\bar{X}}})(\hat{\Gamma}_{B_{\bar{X}}} - \Lambda) - \Gamma^*_{B_{\bar{X}}}(\Gamma^*_{C_{\bar{X}}} - \hat{\Gamma}_{B_{\bar{X}}}) \geq 0$$

and the claim follows from (8). □

5.1.3. *Connectedness of ample Cartier divisors on normal surfaces.* This section provides a connectedness result about ample Cartier divisors on normal surfaces.

Lemma 5.7 — *Let X be a projective normal surface and M an ample Cartier divisor on X. Let $n \in \mathbb{N}_{\geq 2}$ and $C \in |nM|$. Suppose $C = C_1 + C_2$ is a decomposition into two (non-empty) curves. Then*

$$C_1C_2 \geq n - \frac{1}{M^2} \geq n - 1,$$

and $C_1C_2 = n - 1$ if and only if $M^2 = 1$ and one of the C_i is numerically equivalent to M.

Proof. We can numerically write ([Bom73, §4, Lem. 1])

$$C_1 \equiv aM + \varepsilon,\ C_2 \equiv (n-a)M - \varepsilon,$$

where $a = \frac{MC_1}{M^2}$ and $M\varepsilon = 0$.

If $f\colon Y \to X$ is a resolution then, by [Sak84, p. 878], the Picard lattice of Y contains the subspace of f-exceptional curves as a direct summand on which the intersection form is negative definite. Thus the Hodge index theorem on Y implies that the intersection form on X has signiture $(1, k)$ for some $k \geq 0$. Hence $-\varepsilon^2 \geq 0$, with equality if and only if $\varepsilon \equiv 0$.

Since M is an ample Cartier divisor, we have $MC_i > 0$ for $i = 1, 2$, and both of the intersection numbers are integers. Therefore

$$a = \frac{MC_1}{M^2} \geq \frac{1}{M^2}$$

and also

$$\frac{1}{M^2} \leq \frac{MC_1}{M^2} = a = n - \frac{MC_2}{M^2} \leq n - \frac{1}{M^2}.$$

The expression $a(n-a)M^2$, considered as a quadratic function in a, attains its minimum for the smallest (or biggest) possible value of a and thus

$$C_1C_2 = a(n-a)M^2 - \varepsilon^2 \geq n - \frac{1}{M^2} - \varepsilon^2 \geq n - \frac{1}{M^2} \geq n-1. \tag{9}$$

The inequalities in (9) are all equalities if and only if $M^2 = 1$, $\varepsilon \equiv 0$, and $a = aM^2 = 1$ or $a = aM^2 = n-1$. This is possible if and only if one of the curves is numerically equivalent to M. □

5.2. **Proof of Theorem 5.1(*i*).** Let $\xi \subset X$ be a subscheme of length 2. By Theorem 4.1 the 4-canonical map φ_{4I} is a morphism. If $\varphi_{4I}|_\xi$ is an embedding then $\varphi_{mI}|_\xi$ is also an embedding for $m \geq 8$, because $|(m-4)I(K_X+\Delta)|$ is base-point-free, again by Theorem 4.1. If $\varphi_{4I}|_\xi$ is not an embedding then by Lemma 5.4 there exists a well-behaved reduced curve C containing ξ. Corollary 3.4 yields a surjection of global sections

$$H^0(X, \omega_X(\Delta)^{[mI]}) \twoheadrightarrow H^0(C, \omega_X(\Delta)^{[mI]}|_C) \text{ for } m \geq 6.$$

Therefore to show that φ_{mI} $(m \geq 8)$ is an embedding, it suffices to show that $\omega_X(\Delta)^{[mI]}|_C$ defines an embedding for any subscheme ξ of length two that are contracted by φ_{4I}. By Theorem 2.16 it suffices to show that, for any subcurve $B \subset C$, we have $8I(K_X+\Delta)B \geq 2p_a(B)+1$.

By Lemma 5.6, applied to B, we have $(4I+1)(K_X+\Delta)B \geq 2p_a(B)-2$. Since $I(K_X+\Delta)$ is an ample Cartier divisor and thus has degree at least 1 on B, we obtain

$$8I(K_X+\Delta)B \geq 2p_a(B) - 2 + (4I-1)(K_{\bar X} + D_{\bar X} + \Delta_{\bar X})B_{\bar X} \geq 2p_a(B)+1,$$

which concludes the proof.

Remark 5.8 — Employing a trick used below in the proof of Theorem 5.1(*iii*) one could get a better bound of $7I$ for those $\xi \in X$ that are not embedded by φ_{4I}. This does not allow us to conlude that φ_{7I} is very ample in general: let ξ be a subscheme of length two such that $\varphi_{4I}|_\xi$ is an embedding. Then $\varphi_{7I}|_\xi$ is an embedding at ξ unless ξ is supported on a base-point of $3IK_X$.

However, in the latter case we do not know how to find a well-behaved reduced curve in $|3I(K_X+\Delta)|$ or $|4I(K_X+\Delta)|$ containing ξ. This seems to be an artifact of our method and we are led to believe that the bound in Theorem 5.1(*i*) is not sharp.

5.3. **Proof of Theorem 5.1(*iii*) and Theorem 5.2(*i*).** Let $\xi \subset X$ be a subscheme of length 2 and assume we are in the case of Theorem 5.1(iii) or Theorem 5.2(i). Then by Theorem 4.1 the tri-canonical map φ_{3I} is a morphism. If $\varphi_{3I}|_\xi$ is an embedding then $\varphi_{mI}|_\xi$ is also an embedding for $m \geq 6$, because $|(m-3)I(K_X + \Delta)|$ is base-point-free, again by Theorem 4.1. If $\varphi_{3I}|_\xi$ is not an embedding then by Lemma 5.4 there exists a well-behaved reduced curve C containing ξ.

Corollary 3.4 yields a surjection of global sections

$$H^0(X, \omega_X(\Delta)^{[mI]}) \twoheadrightarrow H^0(C, \omega_X(\Delta)^{[mI]}|_C) \text{ for } m \geq 5.$$

Therefore to show that φ_{mI} ($m \geq 6$) is an embedding, it suffices to show that $\omega_X(\Delta)^{[mI]}|_C$ defines an embedding. By Theorem 2.16 it suffices to show that, for any subcurve $B \subset C$, we have $mI(K_X+\Delta)B \geq 2p_a(B)+1$. As $m \geq 5$, this is trivial if $p_a(B) \leq 2$, so we assume $p_a(B) \geq 3$.

By Lemma 5.6, applied to B, we have $(3I+1)(K_X+\Delta)B \geq 2p_a(B)-2$ and thus

$$\begin{aligned} 6I(K_X+\Delta)B &\geq 2p_a(B) - 2 + \frac{3I-1}{3I+1}(2p_a(B)-2) \\ &\geq 2p_a(B) - 2 + \frac{4(3I-1)}{3I+1} \qquad \text{(since } p_a(B) \geq 3) \\ &\geq \begin{cases} 2p_a(B) + \frac{6}{7} & I \geq 2 \\ 2p_a(B) & I = 1 \end{cases}. \end{aligned}$$

Since $6I(K_X+\Delta)B$ is an integer, we have the required $6I(K_X+\Delta)B \geq 2p_a(B)+1$ if $I \geq 2$. If $I = 1$ then $7I(K_X+\Delta)B \geq 2p_a(B)+1$ because $(K_X+\Delta)B \geq 1$. $\square$

5.4. **Proof of Theorem 5.2(*ii*).** For any subscheme $\xi \subset X$ of length two we have a well-behaved curve $C \in |4K_X|$ (not necessarily reduced) containing ξ by Lemma 5.4(ii).

Corollary 3.4 yields a surjection of global sections

$$H^0(X, \omega_X(\Delta)^{[mI]}) \twoheadrightarrow H^0(C, \omega_X(\Delta)^{[mI]}|_C)) \text{ for } m \geq 6.$$

Therefore to show that φ_{mI} ($m \geq 6$) is an embedding, it suffices to show that $\omega_X(\Delta)^{[mI]}|_C$ defines an embedding for any subscheme ξ of length two. By Theorem 2.16 it suffices to show that, for any subcurve $B \subset C$, we have $6I(K_X+\Delta)B \geq 2p_a(B)+1$.

We continue to use the notation from Section 5.1.2. By assumption, $\bar{X}$ is smooth along $D_{\bar{X}}$ and has canonical singularities elsewhere. Thus $\Lambda = \bar{f}^*(K_{\bar{X}}+D_{\bar{X}})-(K_{\bar{Y}}+D_{\bar{Y}})$ is supported on the preimages of the nodes of $D_{\bar{X}}$. On the other hand the divisor $B_{\bar{X}}$, the strict transform of B in the normalisation, is Cartier in a neighbourhood of $D_{\bar{X}}$. The hat transform was defined in terms of intersection numbers, which are defined via the normalisation, and thus $\hat{\Gamma}_{B_{\bar{X}}} - \Gamma^*_{B_{\bar{X}}}$ is trivial on those exceptional divisors mapping to the nodes of $D_{\bar{X}}$. Therefore Λ and $\hat{\Gamma}_{B_{\bar{X}}} - \Gamma^*_{B_{\bar{X}}}$ have disjoint support on $\bar{Y}$ and the intersection number $\Lambda(\hat{\Gamma}_{B_{\bar{X}}} - \Gamma^*_{B_{\bar{X}}}) = 0$.

Lemma 5.5 now implies

$$\begin{aligned} (K_X+\Delta+B)B &\geq 2p_a(B) - 2 + (\Lambda - \hat{\Gamma}_{B_{\bar{X}}} + \Gamma^*_{B_{\bar{X}}})(\hat{\Gamma}_{B_{\bar{X}}} - \Gamma^*_{B_{\bar{X}}}) \\ &\geq 2p_a(B) - 2 - (\hat{\Gamma}_{B_{\bar{X}}} - \Gamma^*_{B_{\bar{X}}})^2 \qquad (10) \\ &\geq 2p_a(B) - 2 \end{aligned}$$

because the intersection form is negative definite on the exceptional divisors of $\bar{f}$.

If $B = C \in |4I(K_X + \Delta)|$ then B is a well behaved Carter divisor and adjunction gives

$$\begin{aligned}6I(K_X+\Delta)B &= (K_X+B)B + \Delta B + (2I-1)(K_X+\Delta)B \\ &\geq 2p_a(B) - 2 + 4(2I-1)I(K_X+\Delta)^2 > 2p_a(B)+1.\end{aligned}$$

If $B < C$ then there is at least one irreducible component $\bar{X}_i$ of $\bar{X}$ such that

$$B_{\bar{X}_i} := B_{\bar{X}} \cap \bar{X}_i < C_{\bar{X}} \cap \bar{X}_i =: C_{\bar{X}_i}.$$

Now Lemma 5.7 says that $C_{\bar{X}_i}B_{\bar{X}_i} - B^2_{\bar{X}_i} \geq 3$. Hence

$$4IK_XB - B^2_{\bar{X}} = C_{\bar{X}}B_{\bar{X}} - B^2_{\bar{X}} \geq C_{\bar{X}_i}B_{\bar{X}_i} - B^2_{\bar{X}_i} \geq 3.$$

Combining with (10), we have

$$6I(K_X+\Delta)B \geq 2p_a(B)+1.$$

As before φ_{6I} embeds C by Theorem 2.16. This implies that φ_{6I} embeds ξ. Since ξ is an arbitrary subscheme of length two, φ_{6I} embeds X. □

5.5. **Proof of Theorem 5.2(*iii*).** The proof is exactly the same as for the previous case with the twist that, under our assumptions, we can choose the curve C to be contained in $|3IK_X|$ by Lemma 5.4. Even though we get weaker connectedness from Lemma 5.7, the numerical criterion is still satisfied for $m \geq 5$. □

5.6. **Proof of Theorem 5.1(*ii*).** Let S be the finite subset of X defined as the union of

$$\{P \in D \,|\, \pi^{-1}(P) \text{ contains a singular point of } \bar{X}\}$$

and

$$\{P \in X \setminus D \,|\, P \text{ is worse than a canonical singularity}\}.$$

Let ξ be the length 2 subscheme consisting of two general points in X. Then there exists a well-behaved curve $C \in |4IK_X|$ containing ξ and choosing C general we can assume that C does not intersect S. Repeating the argument from the proof of Theorem 5.2(*ii*) in Section 5.4 we conclude that $\varphi_{6I}|_C$ is an embedding. Thus φ_{6I} is a morphism that separates every two general points in X, hence is birational. □

6. The index-log-canonical ring

In this section we study the implications of our results so far for the log-canonical ring of a stable log surface. For a log surface (X, Δ) the log-canonical ring is

$$R(X, K_X+\Delta) = \bigoplus_{m\geq 0} H^0(X, \mathcal{O}_X(\lfloor m(K_X+\Delta)\rfloor)).$$

Our results only apply to the index-log-canonical ring which is the Veronese subring

$$R(X, K_X+\Delta)^{(I)} = \bigoplus_{k\geq 0} H^0(X, \omega_X(\Delta)^{[mI]});$$

we do not believe them to be sharp.

Theorem 6.1 — *Let (X, Δ) be a stable log surface of index I.*

(*i*) *The index-log-canonical ring $R(X, K_X+\Delta)^{(I)}$ is generated in degree at most 13.*

(*ii*) *The index-log-canonical ring* $R(X, K_X+\Delta)^{(I)}$ *is generated in degree at most* 10 *if one of the following holds:*
  (a) $I \geq 2$,
  (b) *There is no irreducible component* $\bar X_i$ *of the normalisation such that* $\left(\pi^*(K_X+\Delta)|_{\bar X_i}\right)^2 = 1$ *and the union of* Δ *and the non-normal locus is a nodal curve.*
  (c) X *is normal and we do not have* $I = (K_X+\Delta)^2 = 1$.

Proof. For the first item let L be the ample line bundle $\omega_X^{[4I]}$. Then L is base-point-free by Theorem 4.1. By Corollary 3.4, the line bundle $\omega_X^{[mI]}$ is 0-regular with respect to L for $m \geq 10$. Thus the multiplication map

$$H^0(X, \omega_X^{[mI]}) \otimes H^0(X, \omega^{[4I]}) \to H^0(X, \omega_X^{[(m+4)I]})$$

is surjective for $m \geq 10$ by Mumford's Lemma [Laz04, Thm. 1.8.5], which is also valid for reducible varieties. Thus the ring is generated in degree at most 13, as claimed.

The second item is proved in exactly the same way, noting that under the given conditions, $L = \omega_X^{[3I]}$ is base-point-free and ample and $\omega_X^{[mI]}$ is 0-regular with respect to L for $m \geq 8$. □

Remark 6.2 — One can also deduce from [Mum70, Thm. 3] that the line bundles $\omega_X^{[12I]}$ in case (i) respectively $\omega_X^{[9I]}$ in case (ii) satisfies property N_1, that is, the image of φ_{12I} respectively φ_{9I} is projectively normal and cut out by quadrics.

7. Examples

In this section we construct some examples of stable surfaces and analyse line-bundles on a rational curve with a single 3-multi-node.

We concentrate on examples strictly related to the topic of this article; for further constructions and observations we refer to [LR12a].

Example 7.1 (Very ampleness of $K_{\bar X}+\bar D$ does not descend) — Let $\bar D$ be a smooth plane quartic curve invariant under the involution of $\tau(x,y,z) = (-x,-y,z)$ on $\mathbb{P}^2$; to be concrete set $\bar D = \{f = x^4+y^4+z^4 = 0\}$. Then let X be the (semi-smooth) stable surface corresponding to the triple $(\mathbb{P}^2, \bar D, \tau|_{\bar D})$, that is, we glue $\bar D$ to itself via τ. The quotient $D = \bar D/\tau$ is an elliptic curve and thus by Proposition 2.15 the invariants of X are $K_X^2 = 1$ and $\chi(\mathcal{O}_X) = 3$.

We will now study the canonical ring $R = \bigoplus_k H^0(X, \omega_X^{\otimes k})$ of X and show that while $\pi^*\omega_X^{\otimes k} \cong \mathcal{O}_{\mathbb{P}^2}(k)$ is very ample for $k \geq 1$ the line bundle $\omega_X^{\otimes k}$ is very ample only for $k \geq 5$.

Consider the residue sequence $0 \to \omega_{\mathbb{P}^2}(\bar D)^{\otimes k}(-\bar D) \to \omega_{\mathbb{P}^2}(\bar D)^{\otimes k} \to \omega_{\bar D}^{\otimes k} \to 0$ which gives

$$0 \to H^0(\mathbb{P}^2, \mathcal{O}(k-4)) \to H^0(\mathbb{P}^2, \omega_{\bar X}(\bar D)^{\otimes k}) \xrightarrow{R} H^0(\bar D, \omega_{\bar D}^{\otimes k}) \to 0.$$

It turns out that, if we identify $H^0(\mathbb{P}^2, \omega_{\bar X}(\bar D)^{\otimes k})$ with elements of degree k in $S = \mathbb{C}[x,y,z]$ then the residue of a section is (anti)-invariant if and only if its residue is zero or the section is (anti)-invariant under the induced action of τ on the polynomial ring S. Thus, by Proposition 2.14, $R_k = f \cdot S_{k-4} + S_k^{\pm\tau}$ where $S_k^{\pm\tau}$ are the invariant or anti-invariant polynomials of degree k according to the parity of k.

Writing out the first degrees explicitly it is easy to see that as a subring we have

$$R = \mathbb{C}[x, y, z^2, zf] \subset S,$$

and X is the hypersurface in $\mathbb{P}(1,1,2,5)$ given by the equation $w_4^2 - w_3(w_1^4 + w_2^4 + w_3^2)^2$. In particular, as long as $k \leq 4$ the k-canonical map factors over the quotient $\mathbb{P}^2/\tau = \mathbb{P}(1,1,2)$ and it is very ample for $k \geq 5$. So while $\omega_{\bar{X}}(\bar{D})^{\otimes k}$ is very ample on $\bar{X} = \mathbb{P}^2$ for every $k \geq 1$ this very ampleness does not descend to X.

Incidentially the canonical ring of a smooth surface of general type with $p_g = 2$ and $K^2 = 1$ is known to be of the same form [BHPV04, VII.(7.1)], so we have constructed a surface in the boundary of that irreducible component of the moduli space of smooth surfaces.

Note that this example also shows that our Ansatz to prove base-point-freeness is sharp: the space of sections of $\omega_X^{\otimes k}$ vanishing along the non-normal locus might be empty for $k < 4$.

Example 7.2 (Large K_X^2 is not enough for non-normal surfaces) — For a (connected) stable surface X with canoncial singularities, the bi-canonical map is a morphism as soon as $K_X^2 \geq 5$ and the tri-canonical map is an embedding as soon as $K_X^2 \geq 6$ (see [Cat87]).

We will now construct examples of non-normal stable surfaces (Gorenstein and irreducible) with K_X^2 arbitrarily large but the bi-canonical map not a morphism and the tri-canonical map not an embedding. Morally, the obstructions to being base-point-free as well as the increase of K_X^2 happen locally, so that they cannot affect each other.

Fix once for all an inhomogenous coordinate z on $\mathbb{P}^1$ and let $\tau_0(z) = -z$. On $\bar{X} = \mathbb{P}^1 \times \mathbb{P}^1$ let $H_x = \mathbb{P}^1 \times \{x\}$ and $V_x = \{x\} \times \mathbb{P}^1$ and consider for $k \geq 2$ the divisor

$$\bar{D}_k = H_0 + H_1 + H_\infty + \sum_{j=1}^{k} (V_j + V_{-j}).$$

We specify an involution τ on the normalisation $\bar{D}_k^\nu = H_0 \sqcup H_1 \sqcup H_\infty \sqcup \bigsqcup_{j=1}^k (V_j \sqcup V_{-j})$ by

$$\tau|_{H_0} = \tau_0, \qquad \tau\colon H_1 \xrightarrow{\mathrm{id}} H_\infty,$$
$$\tau\colon V_k \longrightarrow V_{-k},\ z \mapsto \frac{1}{1-z}$$

Because τ preserves the preimages of the nodes of $\bar{D}_k$ it preserves the different $\mathrm{Diff}_{\bar{D}_k^\nu}(0)$ and thus by Theorem 2.13 the triple $(\bar{X}, \bar{D}_k, \tau)$ determines uniquely a stable surface X_k.

We determine the singular points of the non-normal locus as described in Section 4.2: for all $j = 1, \ldots, k$ the points $(\pm j, 0), (\pm j, 1), (\pm j, \infty)$ are mapped to a single point $P_j \in X_k$ and every P_j is a 6-multi-node of D_k. The non-normal locus D_k has $k + 2$ irreducible components: a smooth rational curve containing all P_j, which is the image of H_0, a nodal rational curve with a node at each P_j, which is the image of $H_1 \cup H_\infty$, and for $j = 1, \ldots k$ rational curves $C_j = \pi(V_j \cup V_{-j})$ with a single 3-multi-node at P_j.

The only non-semi-smooth singularities of X_k are degenerate cusps at the points P_j, where X_k locally looks like the cone over a cycle of 6 independent lines. Thus X_k is a Gorenstein stable surface.

We have $\chi(\mathcal{O}_{D_k}) = \chi(\mathcal{O}_{D_k^\nu}) - \chi(\nu_*\mathcal{O}_{D_k^\nu}/\mathcal{O}_{D_k}) = k+2-5k = 2-4k$. On the other hand it is easy to calculate $\chi(\mathcal{O}_{\bar D_k}) = 3-4k$, so by Proposition 2.15 the invariants of X_k are $\chi(\mathcal{O}_{X_k}) = 1 + (2-4k) - (3-4k) = 0$ and $K_{X_k}^2 = (K_{\bar X} + \bar D_k)^2 = 4k-4$.

To prove that $\omega_{X_k}^{\otimes 2}$ has base points and $\omega_{X_k}^{\otimes 3}$ is not an embedding we analyse the restriction the the curves C_j. Its degree is

$$\deg(\omega_{X_k}|_{C_j}) = \frac{1}{2}(K_{\bar X_k} + \bar D_k)(V_j + V_{-j}) = 1.$$

Our claim now follows from the properties of line bundles of low degree on rational curves with a single 3-multi-node analysed in Example 7.3 below.

Example 7.3 (A special curve) — Let B be a rational curve with a single 3-multi-node, $\nu\colon B^\nu \to B$ its normalisation. Then $\chi(\mathcal{O}_B) = \chi(\mathcal{O}_{B^\nu}) - 2 = -1$ and B has arithmetic genus $p_a(B) = 2$.

For any line bundle $\mathcal{L}$ on B the following properties hold.

(i) If $\deg \mathcal{L} \geq 2$ then $H^1(B, \mathcal{L}) = 0$ and $h^0(B, \mathcal{L}) = \deg \mathcal{L} - 1$.
(ii) If $\deg \mathcal{L} = 2$ then $h^0(B, \mathcal{L}) = 1$ and $\mathcal{L}$ does not define a morphism.
(iii) If $\deg \mathcal{L} = 3$ or $\deg \mathcal{L} = 4$ then $|\mathcal{L}|$ is base-point-free but not an embedding.
(iv) If $\deg \mathcal{L} = 4$ then $|\mathcal{L}|$ defines a birational morphism, which is an embedding on the smooth locus.
(v) If $\deg \mathcal{L} \geq 5$ then $|\mathcal{L}|$ defines an embedding.

Proof. By Serre duality $H^1(B, \mathcal{L}) = \mathrm{Hom}_{\mathcal{O}_B}(\mathcal{L}, \omega_B)$. If $H^1(B, \mathcal{L}) \neq 0$, then there is a non-zero $\lambda\colon \mathcal{L} \to \omega_B$. As λ is an isomorphism at the generic point and $\mathcal{L}$ is torsion-free λ is automatically injective and the cokernel is supported on a finite set of points. Thus

$$1 = \chi(\omega_B) \geq \chi(\mathcal{L}) = \deg \mathcal{L} + \chi(\mathcal{O}_B) = \deg \mathcal{L} - 1 \quad \Leftrightarrow \quad \deg \mathcal{L} \leq 2.$$

As $\deg \mathcal{L} \geq 2$ by the assumptions, we have $\deg \mathcal{L} = 2$. Then λ is an isomorphism. On the other hand, since B has a 3-multi-node, ω_B is not locally free — a contradiction. So there is no non-zero λ and $H^1(B, \mathcal{L}) = 0$. This implies the formula for $h^0(B, \mathcal{L})$ by Riemann–Roch and we get (i). The second item is an immediate consequence.

For (iii), note that the embedding dimension of the 3-multi-node is 3 while $\mathcal{L}$ has at most 3 sections so we cannot have an embedding.

Note that, for p a smooth points of B, part (i) applies to $\mathcal{L}(-p)$ so $H^1(B, \mathcal{L}(-p)) = 0$ and p is not a base-point. If p is the 3-multi-node then the ideal sheaf $\mathcal{I}_p$ of p is $\nu_*\mathcal{O}_{B^\nu}(-q_1 - q_2 - q_3)$, so

$$H^1(B, \mathcal{L} \otimes \mathcal{I}_p) = H^1(B^\nu, \nu^*\mathcal{L}(-q_1 - q_2 - q_3)) = 0,$$

because $\nu^*\mathcal{L}(-q_1 - q_2 - q_3))$ is a line-bundle of non-negative degree on $\mathbb{P}^1 \cong B^\nu$. Thus also the 3-multi-nodal point is not a base-point.

If p, p' are smooth points of B then $H^1(B, \mathcal{L}(-p-p')) = 0$ if $\deg \mathcal{L} \geq 4$ by (i) and $\mathcal{L}$ separates smooth points and tangent vectors at smooth points. This proves (iv). The last item follows from Theorem 2.16. □

Consequences of Example 7.3 — *Assume that X is a stable surface such that the non-normal locus D contains a rational curve with a single 3-multi-node. If $\deg IK_X|_B = 1$ then φ_{2I} is not a morphism and φ_{3I} is not an embedding, because the respective restriction to B has this property. In particular, this applies to Example 7.2.*

Appendix A. Curves on surfaces with slc singularities

We fix some notation for this section: let X be a surface with slc singularities and (possibly empty) non-normal locus D. Let $f\colon Y \to X$ be the minimal semi-resolution (Definition 2.11) with conductor divisor D_Y. In some instances when working near $p \in X$ we replace X by a small affine or analytic neighbourhood of p.

Let $\nu\colon \bar{Y} \to Y$ and $\eta\colon \bar{X} \to X$ be the normalisations. We have a commutative diagram

$$\begin{array}{ccc} \bar{Y} & \xrightarrow{\exists\, \bar{f}} & \bar{X} \\ {\scriptstyle\eta}\big\downarrow & & {\scriptstyle\pi}\big\downarrow \\ Y & \xrightarrow{f} & X \end{array} \tag{11}$$

Denote by $D_{\bar{Y}} \subset \bar{Y}$ and $D_{\bar{X}} \subset \bar{X}$) the conductor divisors. Then

(*i*) $K_{\bar{X}} + \bar{D}$ is a $\mathbb{Q}$-Cartier divisor and $K_{\bar{Y}} + D_{\bar{Y}}$ is a Cartier divisor.
(*ii*) $D_{\bar{Y}}$ and D_Y are smooth and $\eta|_{D_{\bar{Y}}}\colon D_{\bar{Y}} \to D_Y$ is a double cover.
(*iii*) D has at most *μ-multi-nodes* (see Definition 4.5), $\bar{D}$ has at most nodes and $\pi|_{\bar{D}}\colon \bar{D} \to D$ is generically two to one.

Now let $B \subset X$ be a well-behaved curve. Our aim is to construct a curve $\hat{B}_Y \subset Y$, the hat transform such that we control the difference of the arithmetic genera $p_a(B)$ and $p_a(\hat{B}_{\bar{Y}})$, where $\hat{B}_{\bar{Y}} \subset \bar{Y}$ is the strict transform of $\hat{B}_Y$. This will be achieved in Proposition A.22. The same idea has been used for surfaces with canonical singularites in [CFHR99], but we have to work harder because our singularites are a lot worse.

Remark A.1 — Let X be a demi-normal surface and $B \subset X$ a well-behaved curve. Recall that for an almost Cartier divisor A on X the sheaf $\mathcal{O}_B(A)$ is the restriction $\mathcal{O}_X(A)|_B$ modulo torsion (see Remark 2.8).

This notation behaves well in the following respect: consider the structure sequence $0 \to \mathcal{O}_X(-B) \to \mathcal{O}_X \to \mathcal{O}_B \to 0$. Tensoring with $\mathcal{O}_X(A)$ we have a natural surjection $\mathcal{O}_X(A) \twoheadrightarrow \mathcal{O}_X(A) \otimes \mathcal{O}_B \twoheadrightarrow \mathcal{O}_B(A)$. The kernel of this surjection is S_2 by the depth lemma and coincides with $\mathcal{O}_X(A-B)$ outside a finite set of points. Thus there is a short exact sequence

$$0 \to \mathcal{O}_X(A-B) \to \mathcal{O}_X(A) \to \mathcal{O}_B(A) \to 0,$$

which generalises the standard sequence we get when A is Cartier.

A.1. Automatic adjunction lemma.

The following technical result will be used several times so we state it here for further reference.

Lemma A.2 — *Let $C \subset X$ be a well-behaved curve and A an almost Cartier divisor on X. Then $H^1(C, \mathcal{O}_C(A)) \neq 0$ if and only if there is a non-empty subcurve $E \subset C$ with a generically onto $\lambda_E\colon \mathcal{O}_E(A) \to \omega_E$. For such a subcurve E, we have $\chi(E, \mathcal{O}_E(A)) \leq \chi(E, \omega_E)$ and equality holds if and only if $\mathcal{O}_E(A) \cong \omega_E$. We can choose E to be connected.*

Proof. By Serre duality, $H^1(C, \mathcal{O}_C(A)) \neq 0$ if and only if there is a non-zero homomorphism $\lambda\colon \mathcal{O}_C(A) \to \omega_C$ in the dual space $\mathrm{Hom}(\mathcal{O}_C(A), \omega_C)$. By automatic adjunction [CFHR99, Lem. 2.4], there is subcurve E of C such that λ restricts to a generically onto $\lambda_E\colon \mathcal{O}_E(A) \to \omega_E$.

On the other hand, if E is a subcurve with a generically onto $\lambda_E\colon \mathcal{O}_E(A) \to \omega_E$, then the composition
$$\mathcal{O}_C(A) \to \mathcal{O}_E(A) \xrightarrow{\lambda_E} \omega_E \hookrightarrow \omega_C$$
is the corresponding non-zero morphism from $\mathcal{O}_C(A)$ to ω_C.

Since $\mathcal{O}_E(A)$ is torsion free, the morphism λ_E is injective. Being generically onto, λ_E has a finite cokernel $\mathcal{Q}$. So we have the following short exact sequence
$$0 \to \mathcal{O}_E(A) \to \omega_E \to \mathcal{Q} \to 0,$$
which yields
$$\chi(E, \omega_E) = \chi(E, \mathcal{O}_E(A)) + \mathrm{length}(\mathcal{Q}) \geq \chi(E, \mathcal{O}_E(A)).$$
This is an equality if and only if the length of $\mathcal{Q}$ is 0 which is in turn equivalent to λ_E being an isomorphism. □

A.2. **Holomorphic Euler characteristics of well-behaved curves.**

Definition A.3 — Let F be a well-behaved curve on the semi-smooth surface Y and $F_{\bar{Y}}$ its strict transform in $\bar{Y}$. We denote by $I_t(F_{\bar{Y}}, D_{\bar{Y}})$ the intersection number of $F_{\bar{Y}}$ and $D_{\bar{Y}}$ at a point $t \in \bar{Y}$.

For a point $q \in Y$ we will define the *local genus correction* $n_q(F)$ of F at q and the *local intersection difference* $d_q(F)$ of F at q such that the relation
$$2n_q(F) + d_q(F) = \sum_{t\in\eta^{-1}(q)} I_t(F_{\bar{Y}}, D_{\bar{Y}})$$
holds. If q is a normal crossing point with preimages t_1, t_2 then
$$n_q(F) = \min\{I_{t_1}(F_{\bar{Y}}, D_{\bar{Y}}), I_{t_2}(F_{\bar{Y}}, D_{\bar{Y}})\},$$
$$d_q(F) = |I_{t_1}(F_{\bar{Y}}, D_{\bar{Y}}) - I_{t_2}(F_{\bar{Y}}, D_{\bar{Y}})|.$$
If q is a pinch point with preimage t then
$$n_q(F) = \lfloor \frac{1}{2} I_{\eta^{-1}(q)}(F_{\bar{Y}}, D_{\bar{Y}}) \rfloor,$$
$$d_q(F) = I_t(F_{\bar{Y}}, D_{\bar{Y}}) - 2\lfloor \frac{1}{2} I_t(F_{\bar{Y}}, D_{\bar{Y}}) \rfloor.$$
At a smooth point $q \in Y$ we set $n_q(F) = d_q(F) = 0$

Remark A.4 — Let F, G be two well-behaved curves on a semi-smooth surface Y. Elementary arithmetics with minimum and round down show that, for $q \in Y_{\mathrm{sing}}$,
$$-\min\{d_q(F), d_q(G)\} \leq n_q(F) + n_q(G) - n_q(F+G) \leq 0$$
and the inequality on the right hand side is an equality if one of F and G is Cartier at q.

We call q a bad point with respect to F and G if $n_q(F) + n_q(G) - n_q(F+G) < 0$; we have $d_q(F), d_q(G) \geq 1$ for bad points q.

We now use these locally defined corrections to prove global identities for the holomorphic Euler-characteristic of well-behaved curves on semi-smooth surfaces.

Proposition A.5 — *Let F and G be well-behaved curves on Y and $F_{\bar{Y}} \subset \bar{Y}$ (resp. $G_{\bar{Y}}$) the strict transforms of F (resp. G). Then*

(*i*) $\chi(Y, \mathcal{O}_X(-F)) = \chi(Y, \mathcal{O}_Y) - \chi(F_{\bar{Y}}, \mathcal{O}_{F_{\bar{Y}}}) + \sum_{q\in Y_{\mathrm{sing}}} n_q(F)$.

(*ii*) $\chi(F, \mathcal{O}_F) = \chi(F_{\bar{Y}}, \mathcal{O}_{F_{\bar{Y}}}) - \sum_{q\in Y_{\mathrm{sing}}} n_q(F)$;

(*iii*) $\chi(G, \mathcal{O}_G(-F)) = \chi(G, \mathcal{O}_G) - FG + \sum_{q\in Y_{\mathrm{sing}}} n_q(F) + n_q(G) - n_q(F+G)$.

Proof. Recall that the map

$$\eta|_{D_{\bar{Y}}} : D_{\bar{Y}} \to D_Y$$

is a double cover between smooth curves; the branch locus of $\eta|_{D_{\bar{Y}}}$ consists exactly of the pinch points of Y. Thus

$$\eta_*\mathcal{O}_{D_{\bar{Y}}} = \mathcal{O}_{D_Y} \oplus \mathcal{L}^{-1}$$

where $\mathcal{L}$ is a line bundle on D_Y with $\mathcal{L}^{\otimes 2} = \mathcal{O}_{D_Y}(\sum_{p \text{ pinch point}} p)$.

There is a commutative diagram of sheaves with exact rows and columns:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \mathcal{O}_Y(-F) \arrow[r] \arrow[d] & \mathcal{O}_Y \arrow[r] \arrow[d] & \mathcal{O}_F \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \eta_*\mathcal{O}_{\bar{Y}}(-F_{\bar{Y}}) \arrow[r] \arrow[d] & \eta_*\mathcal{O}_{\bar{Y}} \arrow[r] \arrow[d] & \eta_*\mathcal{O}_{F_{\bar{Y}}} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{M} \arrow[r] \arrow[d] & \mathcal{L}^{-1} \arrow[r] \arrow[d] & \mathcal{R} \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd} \tag{12}$$

where $\mathcal{M}$ is cokernel of the natural morphism $\mathcal{O}_Y(-F) \to \eta_*\mathcal{O}_{\bar{Y}}(-F_{\bar{Y}})$. Here, because of the Snake Lemma, the last row is exact at $\mathcal{M}$. Using the additivity of the Euler characteristic, we have

$$\begin{aligned}
\chi(Y, \mathcal{O}_X(-F)) &= \chi(Y, \mathcal{O}_Y) - \chi(Y, \mathcal{O}_F) \\
&= \chi(Y, \mathcal{O}_Y) - \chi(Y, \eta_*\mathcal{O}_{F_{\bar{Y}}}) + \chi(Y, \mathcal{R}) \\
&= \chi(Y, \mathcal{O}_Y) - \chi(F_{\bar{Y}}, \mathcal{O}_{F_{\bar{Y}}}) + \chi(Y, \mathcal{R}) \quad \text{(since } \eta \text{ is finite).}
\end{aligned} \tag{13}$$

Note that $\mathcal{R}$ is finite with support in $\hat{B}_Y \cap D_Y$. For (*i*) it suffices to prove the following claim.

Claim. For a point $q \in Y_{\mathrm{sing}}$ we have $\dim_{\mathbb{C}} \mathcal{R}_q = n_q(F)$, where n_q is the local genus correction defined above.

Proof of the claim. We can calculate $\mathcal{R}$ analytically locally around q.

If q is a double normal crossing point of Y, then analytically locally Y is $(xy = 0) \subset \mathbb{C}^3_{x,y,z}$ with $q = (0,0,0)$ and $D_Y = (x = y = 0)$. The normalisation is $\bar{Y} = \mathbb{C}^2_{x,z_1} \sqcup \mathbb{C}^2_{y,z_2}$ and the preimages t_1, t_2 are the origins in the components. The cokernel of inclusion of the coordinate rings $\mathbb{C}[Y] \hookrightarrow \mathbb{C}[\bar{Y}]$ is isomorphic to $\mathbb{C}[D_Y] = \mathbb{C}[z]$. If $F_{\bar{Y}}$ is defined by $f(x, z_1)$ in one irreducible component $\mathbb{C}^2_{x,z_1}$ and by $g(y, z_2)$ in the other irreducible component $\mathbb{C}^2_{y,z_2}$, then the image of its defining ideal $\mathcal{I}_{F_{\bar{Y}}} = (f(x, z_1), g(y, z_2))$ in the localised ring $\mathbb{C}[D_Y]_q = \mathbb{C}[z]_{(z)}$ is the ideal of $\mathbb{C}[z]_{(z)}$ generated by $f(0, z)$ and $g(0, z)$. Note that the orders of $f(0, z)$ and $g(0, z)$ in z is just the intersection numbers $I_{t_i}(F_{\bar{Y}}, D_{\bar{Y}})$, $i = 1, 2$. Therefore we have

$$\begin{aligned}
\dim_{\mathbb{C}} \mathcal{R}_q &= \dim_{\mathbb{C}} \mathbb{C}[z]_{(z)}/(f(0, z), g(0, z)) \\
&= \min\{I_{t_1}(F_{\bar{Y}}, D_{\bar{Y}}), I_{t_2}(F_{\bar{Y}}, D_{\bar{Y}})\}
\end{aligned}$$

If q is a pinch point of Y, then analytically locally Y is $(x^2 - y^2 z = 0) \subset \mathbb{C}^3$ with $p = (0,0,0)$ and $D_Y = (x = y = 0)$. The normalisation $\bar{Y}$ of Y is $\mathbb{C}^2_{u,y}$ with normalisation map

$$\begin{array}{ccc} \bar{Y} & \to & Y \\ (u,y) & \mapsto & (uy, y, u^2). \end{array}$$

The preimage t of q is the origin in $\bar{Y} = \mathbb{C}^2_{u,y}$ and the conductor divisor $D_{\bar{Y}}$ is defined by $y = 0$ in $\bar{Y}$. The cokernel of the inclusion of coordinate rings $\mathbb{C}[Y] \hookrightarrow \mathbb{C}[\bar{Y}]$ is naturally isomorphic to $u\mathbb{C}[D_Y] = u\mathbb{C}[z]$. (Note that in $\mathbb{C}[\bar{Y}]$ we have $z = u^2$.) Now suppose $F_{\bar{Y}}$ is defined by $f(u,y) = 0$ locally around $q = (0,0)$. Then $\mathcal{I}_{F_{\bar{Y}}}$ is the ideal of $\mathbb{C}[u,y]$ generated by $f(u,y)$. Note that the order of $f(u,0)$ is just the intersection number $I_t(F_{\bar{Y}}, D_{\bar{Y}})$. Then the image of $\mathcal{I}_{F_{\bar{Y}}}$ in the localised module $u\mathbb{C}[z]_{(z)}$ is a submodule generated by uz^k, where $\dim_{\mathbb{C}} \mathcal{R}_q = k = \lfloor \frac{1}{2} I_t(F_{\bar{Y}}, D_{\bar{Y}}) \rfloor$. This concludes the proof. □

The first row of (12) gives $\chi(F, \mathcal{O}_F) = \chi(Y, \mathcal{O}_Y) - \chi(Y, \mathcal{O}_Y(-F))$, so (ii) is implied by (i).

The short exact sequence $0 \to \mathcal{O}_Y(-F-G) \to \mathcal{O}_Y(-F) \to \mathcal{O}_G(-F) \to 0$ gives $\chi(G, \mathcal{O}_G(-F)) = \chi(Y, \mathcal{O}_Y(-F)) - \chi(Y, \mathcal{O}_Y(-F-G))$. Applying (i) to both terms on the right hand side and then substituting (ii) gives (iii). □

A.3. **Resolution graphs and semi-rationality.** We will now recall some more of the classification of slc singularities. The resolution graphs of log-canonical surface singularities are well known (e.g. [KM98b, Ch. 4]) so we concentrate on the non-normal case; our sources are [Kol10, 17] and [KSB88, Sect. 4].

Over a non-normal point $p \in X$ we can write $\bar{Y}$ analytically locally $\bar{Y} = \cup \bar{Y}_\alpha$ as the union of local irreducible components. On each component the f-exceptional divisors together with the components of the double locus give rise to an (extended) dual graph: every f-exceptional component, which are all rational because we are over a non-normal point of X, gives a vertex which is either marked with "$\circ$" or with the negative self-intersection; we add a "$\bullet$" for every component of the conductor divisor $D_{\bar{Y}}$ and connect two vertices if the corresponding curves intersect.

The edges connecting the resolution graph to the boundary components are marked with the coefficent of the different $\mathrm{Diff}_{D_{\bar{X}}}(0)$ at the corresponding point of the conductor divisor on $\bar{X}$.

The following three cases can occur:

(C1) $$\bullet \overset{1-\frac{1}{\delta}}{\text{——}} c_1 - \cdots - c_n \qquad (c_i \geq 2)$$

where δ is the determinant of the intersection form of the exceptional divisors.

(C2) $$\bullet \overset{1}{-} c_1 - \cdots - c_n \overset{1}{-} \bullet \qquad (c_i \geq 1)$$

and if some $c_j = 1$, then $n = 1$, because we consider the minimal semi-resolution.

(Dh) $$\begin{array}{llll} & & & 2 \\ & & & / \\ \bullet \overset{1}{-} c_1 - \cdots - & c_n & & \qquad (n \geq 2,\ c_i \geq 2). \\ & & & \backslash \\ & & & 2 \end{array}$$

According the the type of extended dual graph associated to the curves in an irreducible component $\bar{Y}_\alpha$ we say $Y_\alpha := \eta(\bar{Y}_\alpha)$ is of type (C1) (resp. (C2), (Dh)).

The whole extended dual graph of an slc singularity is obtained by attaching graphs of the types (C1), (C2), and (Dh) along the boundary components, with the restriction that the differents should match (see Theorem 2.13). There can be *free* boundary components in the resolution graph if the exceptional divisor intersects the conductor on Y in a pinch point (compare [KSB88, Prop. 4.27] for more details). Thus the exceptional divisors form a tree of rational curves unless we glue a number of components of type (C2) in a circle, that is, the singularity is a degenerate cusp.

The following is an important property of a singularity.

Definition A.6 ([KSB88, Def. 4.14], [vS87, Def. 4.1.1]) — If $R^1 f_*\mathcal{O}_Y = 0$ then we say X has semi-rational singularities.

Morally all results valid for rational singularies hold also in the semi-rational case. The following makes the connection to slc singularities.

Lemma A.7 — *Let $p \in X$ be a slc surface singularity and $f\colon Y \to X$ the minimal semi-resolution.*

The point p is not semi-rational if and only if p is simple elliptic, a cusp, or a degenerate cusp. In this case $\dim_{\mathbb{C}}(R^1 f_*\mathcal{O}_Y)_p = 1$. (This number is sometimes called the geometric genus of a singularity.)

Proof. In the normal case this is well known, see for example [Kaw88, Lem. 9.3]. In lack of an appropriate reference we sketch a proof in the non-normal case.

The statement that $\dim_{\mathbb{C}}(R^1 f_*\mathcal{O}_Y)_p = 1$ for a degenerate cusp is contained in [vS87, Thm. 4.3.6]. So it remains to prove that in all other cases the singularity is semi-rational.

Suppose first $p \in X$ is a non-normal Gorenstein point but not a degenerate cusp. By [KSB88, Thm. 4.21] p is a normal crossing or a pinch point and thus semi-rational. So it remains to show that a non-Gorenstein slc singularity $p \in X$ is semi-rational.

Let $\tilde{p} \in \tilde{X}$ be the canonical index one cover of $p \in X$. Then $\tilde{X} \to X$ is a G-covering branched only at the single point p ([KSB88, Thm. 4.24]) for some finite group G.

If $\tilde{p} \in \tilde{X}$ is semi-canonical, i.e., a normal crossing or pinch point, then we can argue as in [Kov00, Thm. 1], which works for semi-rationality as well.

Otherwise $\tilde{p} \in \tilde{X}$ is a denegerate cusp. Let $\tilde{g}\colon \tilde{W} \to \tilde{X}$ the minimal semi-resolution. Then the G-action lifts to $\tilde{W}$ and we have a commutative diagram

$$\begin{tikzcd}
\tilde{W} \arrow[r, "\lambda"] \arrow[d, "\tilde{g}"'] & W := \tilde{W}/G \arrow[d, "g"] \\
\tilde{X} \arrow[r, "\varphi"] & X = \tilde{X}/G
\end{tikzcd}$$

Argueing again as in [Kov00, Thm. 1], we see that W has semi-rational singularities.

Denote by φ_*^G (resp. λ_*^G) be the composite functor of pushforward and taking the G-invariant part. Then φ_*^G (resp. λ_*^G) is an exact funtor from the category of G-equivariant $\mathcal{O}_{\tilde{X}}$-modules (resp. $\mathcal{O}_{\tilde{W}}$-modules) to the category of $\mathcal{O}_X$-modules (resp. $\mathcal{O}_W$-modules). Then $g_* \circ \lambda_*^G = \varphi_*^G \circ g$, and by the Grothendieck spectral sequence we get

$$R^1 g_*\mathcal{O}_W \cong R^1(g_* \circ \lambda_*^G)(\mathcal{O}_{\tilde{W}}) \cong R^1(\varphi_*^G \circ \tilde{g})(\mathcal{O}_{\tilde{W}}) \cong \varphi_*^G R^1\tilde{g}_*(\mathcal{O}_{\tilde{W}}) \cong (R^1\tilde{g}_*(\mathcal{O}_{\tilde{W}}))^G.$$

With the same argument as in [Kaw88, Proof of Thm. 9.6, p. 143] one proves that G acts effectively on $R^1\tilde{g}_*(\mathcal{O}_{\tilde{W}}) \cong \mathbb{C}$ thus $R^1\tilde{g}_*(\mathcal{O}_{\tilde{W}}))^G = 0$ and $p \in X$ is semi-rational also in this case. This concludes the proof. □

An alternative approach to this result is to compute the fundamental cycle on a stable improvement in the sense of [vS87] and then use [vS87, Thm. 4.1.3].

Remark A.8 — Let $p \in X$ be a slc point and Y the semi-resolution of a sufficiently small neighbourhood of p. Then for every effective divisor E supported on the exceptional divisors we have a surjection

$$(R^1f_*\mathcal{O}_Y)_p \cong H^1(Y, \mathcal{O}_Y) \twoheadrightarrow H^1(E, \mathcal{O}_E).$$

Thus if p is semi-rational then $h^1(E, \mathcal{O}_E) = 0$ and $\chi(E, \mathcal{O}_E) = h^0(E, \mathcal{O}_E) \geq 1$; if p is not semi-rational then $h^1(E, \mathcal{O}_E) \leq 1$ and $\chi(E, \mathcal{O}_E) \geq h^0(E, \mathcal{O}_E) - 1 \geq 0$.

More precisely, in the non-semi-rational case equality can only occur if the support of E is the full exceptional locus since otherwise E is supported on the exceptional divisor of a semi-rational singularity.

A.4. **Semi-numerical cycle.** .

Definition A.9 — Let $p \in X$ be a non-semi-smooth point and E_i the exceptional divisors over p. The semi-numerical cycle Z over p is a minimal Weil divisor $Z = \sum_i a_iE_i$ of Y such that

(i) $a_i \in \mathbb{Z}$ for any i;

(ii) $(Z_{\bar{Y}} + D_{\bar{Y}})E_{i,\bar{Y}} \leq 0$ for any i.

Remark A.10 — If X is normal, then $D_{\bar{Y}}$ is empty and Definition A.9 coincides with the usual definition of a numerical cycle ([Rei97, Sect. 4.5]); the existence and the uniqueness of a semi-numerical cycle is proved in the same fasion as for the normal singularities, using the negative definiteness of the intersection form on the exceptional curves. The semi-numberical cycle turns out to carry the cohomology $R^1f_*\mathcal{O}_Y$ (cf. Remark A.13).

See [vS87, 3.4] for a discussion of the notion of fundamental cycle for a more general class of non-normal surfaces singularites.

In the case where $p \in X$ is normal, the numerical cycle is nicely elaborated in [Rei97, Section 4] (see also [KM98b, Thm. 4.7]), so we concentrate on the non-normal case.

Remark A.11 (Semi-numerical cycle on non-normal slc singularities) — Let $p \in X$ be a non-normal slc point. Locally analytically around the preimage of p we decompose the resolution Y into irreducible components of the type presented in Section A.3 and correspondingly the semi-numerical cycle $Z = \bigcup_\alpha Z_\alpha$ where $Z_\alpha \subset Y_\alpha$. Since the intersection form is defined via the normalisation the divisors Z_α are uniquely determined by the configuration of exceptional curves and boundary components on Y_α.

Computing in each of the different cases we see that Z_α is the reduced sum of exceptional divisors except in the the following cases:

(i) The component Y_α is of type (C2) with extended dual graph

$$\bullet \;-\; 1 \;-\; \bullet$$

and $Z_\alpha = 2E$, where E is the exceptional curve.

(*ii*) The component is of type (Dh) with dual graph

$$\begin{array}{ccccccc} & & & & & & 2 \\ & & & & & \diagup & \\ \bullet - 2 & - & \cdots & - & 2 & & \\ & & & & & \diagdown & \\ & & & & & & 2 \end{array}$$

and denoting by E' and E'' the two exceptional curves on the right of the fork and by E_j the exceptional curve in the chain to the left of the fork the restriction of the semi-numerical cycle is $Z_\alpha = E' + E'' + 2\sum_{j=1}^n E_j$.

In particular, Z has multiplicity at most 2 at each irreducible exceptional curves.

The next result shows the importance of the semi-numerical cycle for the computation of higher pushforward sheaves.

Lemma A.12 — *Let C and F be well-behaved curves on X and Y respectively such that $f_*F = C$ as Weil divisors. Suppose moreover $FE \le 0$ for any effective exceptional divisor E over p. Then $R^1 f_*\mathcal{O}_Y(-F)_p \cong H^1(Z, \mathcal{O}_Z(-F))$ where Z is the semi-numerical cycle over p.*

Remark A.13 — Applying the above to an empty curve we see that $H^1(Z, \mathcal{O}_Z) = (R^1 f_*\mathcal{O}_Y)_p$ where $p \in X$ and Z is the semi-numerical cycle over p.

Proof. If p is semi-smooth then the map f is finite in a neighbourhood of p and both sides are 0. So assume p to be a non-semi-smooth point of X. Let E be any effective divisor supported on the exceptional locus over p. We have a commutative diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
 & \mathcal{O}_Y(-Z-E-F) \arrow[r, equal] \arrow[d] & \mathcal{O}_Y(-Z-E-F) \arrow[d] & & \\
0 \arrow[r] & \mathcal{O}_Y(-F-Z) \arrow[r] \arrow[d] & \mathcal{O}_Y(-F) \arrow[r] \arrow[d] & \mathcal{O}_Z(-F) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathcal{O}_E(-F-Z) \arrow[r] \arrow[d] & \mathcal{O}_{Z+E}(-F) \arrow[r] \arrow[d] & \mathcal{O}_Z(-F) \arrow[r] & 0 \\
 & 0 & 0 & &
\end{tikzcd}$$

where the exactness of the columns and the middle row follows from Remark A.1. Hence also the last row is exact and we get an exact sequence in cohomology:

$$H^1(E, \mathcal{O}_E(-Z-F)) \to H^1(Z+E, \mathcal{O}_{Z+E}(-F)) \to H^1(Z, \mathcal{O}_Z(-F)) \to 0.$$

Claim. $H^1(E, \mathcal{O}_E(-Z-F)) = 0$.

Proof of the claim. Suppose on the contrary that $H^1(E, \mathcal{O}_E(-Z-F)) \neq 0$. By Lemma A.2, there is a subcurve $E' \subset E$ such that

$$\chi(E', \mathcal{O}_{E'}(-Z-F)) \le \chi(E', \omega_{E'}) = -\chi(E', \mathcal{O}_{E'}). \tag{14}$$

Recall that by Remark A.8

$$\begin{aligned} &\chi(E', \mathcal{O}_{E'}) \ge 0 \text{ and} \\ &\chi(E', \mathcal{O}_{E'}) \ge 1 \text{ unless } p \text{ is not semi-rational and } \operatorname{supp} E' = f^{-1}(p). \end{aligned} \tag{15}$$

Applying Proposition A.5(iii) to $\mathcal{O}_{E'}(-Z-F)$ equation (14) becomes

$$2\chi(E', \mathcal{O}_{E'}) + \sum_{q\in Y_{\mathrm{sing}}} n_q(E') + n_q(Z+F) - n_q(E'+Z+F) \le (Z+F)E' \tag{16}$$

If $p \in X$ is normal, then

$$\sum_{q\in Y_{\mathrm{sing}}} n_q(E') + n_q(Z+F) - n_q(E'+Z+F) = 0$$

and hence

$$2\chi(E', \mathcal{O}_{E'}) \le (Z+F)E' \le 0.$$

If $p \in X$ is rational or $\mathrm{supp}(E') \ne f^{-1}(p)$, then $2\chi(E', \mathcal{O}_{E'}) \ge 2$ by (15) — a contradiction. If $p \in X$ is not rational and $\mathrm{supp}(E') = f^{-1}(p)$ then $E' = Z + E''$ for some effective E''. Hence $(Z+F)E' \le ZE' = Z^2 + ZE'' \le Z^2 < 0$ while $2\chi(E', \mathcal{O}_{E'}) \ge 0$ which again contradicts (15).

Now we can assume $p \in X$ is non-normal. Let $E'_{\bar{Y}} \subset \bar{Y}$ be the strict transform of E'. Then

$$n_q(Z+F) - n_q(E'+Z+F) \ge -\sum_{t\in\eta^{-1}(q)} I_t(E'_{\bar{Y}}, D_{\bar{Y}}) \ge ZE' \tag{17}$$

where the last inequality is because of the definition of the semi-numerical cycle. Combining (16) and (17), we have

$$2\chi(E', \mathcal{O}_{E'}) + \sum_{q\in Y_{\mathrm{sing}}} n_q(E') \le FE' \le 0.$$

where the last inequality is by our assumption on F.

If $p \in X$ is non-semi-rational with $\mathrm{supp}(E') = f^{-1}(p)$, then $\sum_{q\in Y_{\mathrm{sing}}} n_q(E') > 0$ and $\chi(E', \mathcal{O}_{E'}) \ge 0$ by (15) — contradiction. Otherwise $\chi(E', \mathcal{O}_{E'}) \ge 1$ and again we get a contraction.

Thus a subcurve E' as in (14) cannot exist and $H^1(E, \mathcal{O}_E(-Z-F)) = 0$ as claimed. □

Therefore $H^1(Z, \mathcal{O}_Z(-F)) = H^1(Z+E, \mathcal{O}_{Z+E}(-F))$ for every effective exceptional divisor E over p. Moreover, the surjection $\mathcal{O}_Y(-F)|_{Z+E} \twoheadrightarrow \mathcal{O}_{Z+E}(-F)$ induces an isomorphism $H^1(Z+E, \mathcal{O}_{Z+E}(-F)) = H^1(Z+E, \mathcal{O}_Y(-F)|_{Z+E})$ because the kernel is supported on points. By the theorem on formal functions ([Har77, Thm. III.11.1]) we have $R^1 f_*(\mathcal{O}_Y(-F))_p = H^1(Z, \mathcal{O}_Z(-F))$ as claimed. □

We now give some lower bounds on the Euler characteristic of subcurves of semi-numerical cycles.

Lemma A.14 — *Let Z be the semi-numerical cycle over $p \in X$ and $E \subset Z$ a connected subcurve. Then*

$$2\chi(E, \mathcal{O}_E) \ge \sum_{q\in Y_{\mathrm{sing}}} d_q(E)$$

with equality if and only if E satisfies one of the following

(i) *$p \in X$ is simple elliptic singularity or a cusp and $E = Z$.*

(ii) *$p \in X$ is non-normal, E is reduced and every connected component of $\bar{E}$ is a chain of smooth rational curves intersecting $D_{\bar{Y}}$ in two points.*

Proof. If $p \in X$ is normal, then $d_q(E) = 0$ for all q and the assertions follow from Remark A.8. So in the following we assume $p \in X$ is non-normal and that E is connected.

Denote by $\bar{E}$ the strict transform of E in $\bar{Y}$. Let $\bar{E}^{(\alpha)}$ $(1 \le \alpha \le n)$ be the connected components of $\bar{E}$ and $E^{(\alpha)} := \eta_* \bar{E}^{(\alpha)}$ the pushforward as Weil divisors. We arrange the labels in such a way that $E^{(\alpha)} \cap E^{(\alpha+1)} \neq \emptyset$ for $1 \le \alpha \le n-1$. By Proposition A.5, we have

$$\chi(E, \mathcal{O}_E) = \chi(\bar{E}, \mathcal{O}_{\bar{E}}) - \sum_{q \in Y_{\mathrm{sing}}} n_q(E) = \sum_{\alpha=1}^{n} \chi(\bar{E}^{(\alpha)}, \mathcal{O}_{\bar{E}^{(\alpha)}}) - \sum_{q \in Y_{\mathrm{sing}}} n_q(E), \tag{18}$$

and similarly

$$\chi(E_{\mathrm{red}}, \mathcal{O}_{E_{\mathrm{red}}}) = \sum_{\alpha=1}^{n} \chi(\bar{E}^{(\alpha)}_{\mathrm{red}}, \mathcal{O}_{\bar{E}^{(\alpha)}_{\mathrm{red}}}) - \sum_{q \in Y_{\mathrm{sing}}} n_q(E_{\mathrm{red}}). \tag{19}$$

First we look at the reduction E_{red} of E, which we assumed to be connected. If $p \in X$ is either semi-rational or non-semi-rational but $\mathrm{supp}(E) \neq f^{-1}(p)$, then E_{red} is a reduced tree of rational curves; we have $\chi(E_{\mathrm{red}}, \mathcal{O}_{E_{\mathrm{red}}}) = 1$ and since a local intersection difference can only occur at the end points $\sum_{q \in Y_{\mathrm{sing}}} d_q(E_{\mathrm{red}}) \le 2$.

If $p \in X$ is a non-semi-rational non-normal point then it is a degenerate cusp, and if $\mathrm{supp}(E) = f^{-1}(p)$ the divisor E_{red} is a cycle of rational curve so that $\chi(E_{\mathrm{red}}, \mathcal{O}_{E_{\mathrm{red}}}) = 0$ and $\sum_{q \in Y_{\mathrm{sing}}} d_q(E_{\mathrm{red}}) = 0$. In both cases we have

$$2\chi(E_{\mathrm{red}}, \mathcal{O}_{E_{\mathrm{red}}}) - \sum_{q \in Y_{\mathrm{sing}}} d_q(E_{\mathrm{red}}) \ge 0 \tag{20}$$

with equality if and only if E_{red} is as described in (ii).

In general, E is obtained from E_{red} by adding some irreducible components of E_{red}. More precisely, let $F_1, \dots, F_k$ be the irreducible components of $E - E_{\mathrm{red}}$ so that we can write $E = E_{\mathrm{red}} + \sum_{1 \le j \le k} F_j$. We order in such a way that $F_1, \dots, F_{k'}$ have non-empty intersection with D_Y while $F_{k'+1}, \dots, F_k$ do not intersect D_Y.

Using the computation of semi-numerical cycles from Remark A.11 we distinguish three possible cases for $F_j \subset Y_{(\alpha_j)}$.

(a) F_j is a (-1)-curve. Then $E^{(\alpha_j)} = 2F_j$ because the only possibility is type (C2) of length 1. By the adjunction formula on $\bar{Y}$, we have $\chi(\bar{E}^{(\alpha_j)}, \mathcal{O}_{\bar{E}^{(\alpha_j)}}) = 3$. Also $\sum_{q \in Y_{\mathrm{sing}}} \sum_{t \in \eta^{-1}(q)} I_t(\bar{F}_j, D_{\bar{Y}}) = 2$ for such a curve F_j.

(b) F_j lies in an irreducible component $Y_{\alpha_j} \subset Y$ of type (Dh) and intersects D_Y. Write $\bar{E}^{(\alpha_j)} = \bar{E}^{(\alpha_j)}_1 + \bar{F}_j$. Computing Euler characteristics for the structure sequence of $\bar{E}^{(\alpha_j)}_1 \subset \bar{E}^{(\alpha_j)}$ in the explicit situation of Remark A.11 one obtains that $\chi(\bar{E}^{(\alpha_j)}, \mathcal{O}_{\bar{E}^{(\alpha_j)}}) \ge 2$. For such an F_j, we have $\sum_{q \in Y_{\mathrm{sing}}} \sum_{t \in \eta^{-1}(q)} I_t(\bar{F}_j, D_{\bar{Y}}) = 1$.

(c) F_j lies in an irreducible component $Y_\alpha \subset Y$ of type (Dh) and but there is no non-reduced irreducible component of E_α intersecting the conductor. Thus $\bar{E}^{(\alpha_j)}$ is supported on the exceptional divisor of a rational surface singularity and $\chi(\bar{E}^{(\alpha_j)}, \mathcal{O}_{\bar{E}^{(\alpha_j)}}) \ge 1$ by [Rei97, Prop. 4.12].

Let $r_1 := \#\{j \,|\, 1 \leq j \leq k', F_j \text{ is a } (\text{-}1)\text{-curve}\}$ and $r_2 := k' - r_1$. Then since by classification $\chi(\bar{E}^{(\alpha)}_{\mathrm{red}}, \mathcal{O}_{\bar{E}^{(\alpha)}_{\mathrm{red}}}) = 1$ for every connected component of $\bar{E}$ in total we get

$$2\sum_{\alpha=1}^{n} \chi(\bar{E}^{(\alpha)}, \mathcal{O}_{\bar{E}^{(\alpha)}}) - 2\sum_{\alpha=1}^{n} \chi(\bar{E}^{(\alpha)}_{\mathrm{red}}, \mathcal{O}_{\bar{E}^{(\alpha)}_{\mathrm{red}}}) \geq 2(2r_1 + r_2). \tag{21}$$

On the other hand, we have

$$\begin{aligned}
&\sum_{q\in Y_{\mathrm{sing}}} 2n_q(E_{\mathrm{red}}) + d_q(E_{\mathrm{red}}) - \sum_{q\in Y_{\mathrm{sing}}} 2n_q(E) + d_q(E)\\
=&\bar{E}_{\mathrm{red}} D_{\bar{Y}} - \bar{E} D_{\bar{Y}} \qquad \text{(by Definition A.3)}\\
=&-\sum_{j=1}^{k'} \bar{F}_j D_{\bar{Y}}\\
=&-(2r_1 + r_2)
\end{aligned}$$

Adding these equations and using (18), (19) and (20), we have

$$2\chi(E, \mathcal{O}_E) - \sum_{q\in Y_{\mathrm{sing}}} d_q(E) \geq 2\chi(E_{\mathrm{red}}, \mathcal{O}_{E_{\mathrm{red}}}) - \sum_{q\in Y_{\mathrm{sing}}} d_q(E) + 2r_1 + r_2 \geq 2r_1 + r_2 \geq 0.$$

If equality holds then $r_1 = r_2 = 0$ and $2\chi(E_{\mathrm{red}}, \mathcal{O}_{E_{\mathrm{red}}}) = \sum_{q\in Y_{\mathrm{sing}}} d_q(E)$ so E_{red} is as described in (*ii*). As a conseqence, no irreducible component of E_{red} is contained in a component of type (Dh) and there cannot be non-reduced irreducible components not intersecting the conductor. Thus $E = E_{\mathrm{red}}$ and E is as in (*ii*).

On the other hand it is easy to see that equality holds if E is as in (*ii*). □

Lemma A.15 — *Let $F \subset Y$ be a well-behaved curve such that $FE_i \leq 0$ for any exceptional curve E_i over $p \in X$. Then we have*

$$\dim_{\mathbb{C}} R^1 f_* \mathcal{O}_Y(-F)_p \leq \dim_{\mathbb{C}} (R^1 f_* \mathcal{O}_Y)_p + \frac{1}{2} \# \left\{ q \in f^{-1}(p) \,|\, d_q(F) > 0 \right\},$$

where the function d_q is as in Definition A.3.

Proof. If $p \in X$ is semi-smooth then the map f is an isomorphism in a neighbourhood of p and $\dim_{\mathbb{C}} R^1 f_* \mathcal{O}_Y(-F)_p = 0$. So we may assume that p is a non-semi-smooth singularity of X. Let $Z \subset Y$ the semi-numerical cycle over p. By Lemma A.12 we have

$$R^1 f_*(\mathcal{O}_Y(-F))_p = H^1(Z, \mathcal{O}_Z(-F)).$$

and it remains to estimate the dimension of the right hand side.

Suppose $H^1(Z, \mathcal{O}_Z(-F)) \neq 0$. Then, by Lemma A.2, there is a connected subcurve $E \subset Z$ such that $\chi(E, \mathcal{O}_E(-F)) \leq \chi(E, \omega_E) = -\chi(E, \mathcal{O}_E)$ with equality if and only if $\mathcal{O}_E(-F) \cong \omega_E$. Combining with Proposition A.5(*iii*) yields

$$2\chi(E, \mathcal{O}_E) + \sum_{q\in Y_{\mathrm{sing}}} n_q(E) + n_q(F) - n_q(E + F) \leq FE, \tag{22}$$

with equality if and only if $\mathcal{O}_E(-F) \cong \omega_E$. By Remark A.4

$$n_q(E) + n_q(F) - n_q(E + F) \geq -\min\{d_q(E), d_q(F)\} \geq -d_q(E)$$

and hence

$$2\chi(E, \mathcal{O}_E) + \sum_{q\in Y_{\mathrm{sing}}} n_q(E) + n_q(F) - n_q(E + F) \geq 2\chi(E, \mathcal{O}_E) - \sum_{q\in Y_{\mathrm{sing}}} d_q(E) \geq 0$$

where the last inequality comes from Lemma A.14. Since $FE \leq 0$ by assumption we have equality in (22) and $\mathcal{O}_E(-F) \cong \omega_E$. This implies $FE = 0$,

$$2\chi(E, \mathcal{O}_E) = \sum_{q \in Y_{\text{sing}}} d_q(E),$$

and $n_q(E) + n_q(F) - n_q(E+F) = -\min\{d_q(E), d_q(F)\} = -d_q(E)$ for all $q \in Y_{\text{sing}}$. In particular,

$$d_q(F) \geq d_q(E). \tag{23}$$

Case 1: $p \in X$ is normal. By Lemma A.14, $p \in X$ is either a simple elliptic singularity or a cusp and $E = Z$. In particular

$$h^1(Z, \mathcal{O}_Z(-F)) = h^1(Z, \omega_Z) = h^1(Z, \mathcal{O}_Z) = 1,$$

and we have equality in the claim of the Lemma.
Case 2: $p \in X$ is non-normal. Let $\mathcal{E}$ be set of connected subcurves E of Z such that there is a generically onto homomorphism $\lambda_E \colon \mathcal{O}_E(-F) \to \omega_E$.

Claim: If $E \neq E' \in \mathcal{E}$ then E and E' have disjoint support.

Proof. Interpreting the morphisms $\lambda \colon \mathcal{O}_E(-F) \to \omega_E$ and $\lambda' \colon \mathcal{O}_{E'}(-F) \to \omega_{E'}$ as elements in $H^1(Z, \mathcal{O}_Z(-F))$ a general linear combination will give a generically onto morphism supported on $E \cup E'$. Thus our claim follow if we can show that for a curve $E \in \mathcal{E}$ no connected proper subcurve can be contained in $\mathcal{E}$.

By Lemma A.14, E lies completely in the union of irreducible components of type (C2), so E is a reduced nodal curve of arithmetic genus 0 or 1. By the above $\mathcal{O}_E(-F) \cong \omega_E$. For every connected proper subcurve $E' \subset E$ we have $\deg \mathcal{O}_E(-F)|_{E'} = \deg \omega_E|_{E'} \geq -1 > -2 = \deg \omega_{E'}$ and thus there is no generically onto morphism from $\mathcal{O}_{E'}(-F)$ to $\omega_{E'}$. □

Since different curves in $\mathcal{E}$ are disjoint, we have

$$H^1(Z, \mathcal{O}_Z(-F)) = \bigoplus_{E \in \mathcal{E}} H^1(E, \mathcal{O}_E(-F)) = \bigoplus_{E \in \mathcal{E}} H^1(E, \omega_E) \cong \mathbb{C}^{\#\mathcal{E}}. \tag{24}$$

If $p_a(E) = 1$ for some $E \in \mathcal{E}$ then p is a degenerate cusp and E is the reduced preimage of p. Thus $\mathcal{E} = \{E\}$ and $H^1(Z, \mathcal{O}_Z(-F)) = H^1(E, \omega_E)$ and the claimed inequality holds.

Otherwise by Lemma A.14, every $E \in \mathcal{E}$ is a chain of rational curves such that the end(s) of the chain intersect the conductor in two (different) points q_1 and q_2; at these intersection points q_i we have $1 = d_{q_i}(E) \leq d_{q_i}(F)$ by (23). Since every two different curves in $\mathcal{E}$ are disjoint the following inequality holds

$$\#\mathcal{E} \leq \frac{1}{2} \#\{q \in f^{-1}(p) \mid d_q(F) > 0\}.$$

Together with equation (24) and Lemma A.12 this completes the proof of the lemma. □

A.5. **A relative duality.** As a preparation for the relative duality result we need the following lemma.

Lemma A.16 — *Let C be a well-behaved curve and A an almost Cartier divisor on X. Then* $\operatorname{Ext}^1_{\mathcal{O}_X}(\mathcal{O}_C, \omega_X(A)) \cong \operatorname{Ext}^1_{\mathcal{O}_X}(\mathcal{O}_C(-A), \omega_X)$.

Proof. We look at the structure sequence
$$0 \to \mathcal{O}_X(-C) \to \mathcal{O}_X \to \mathcal{O}_C \to 0$$
and, applying $\mathscr{H}om_{\mathcal{O}_X}(\cdot, \omega_X(A))$, obtain an exact sequence
$$0 \to \omega_X(A) \to \mathscr{H}om_{\mathcal{O}_X}(\mathcal{O}_X(-C), \omega_X(A)) \to \mathscr{E}xt^1_{\mathcal{O}_X}(\mathcal{O}_C, \omega_X(A)) \to 0.$$
Since $\mathcal{O}_X(A)$ is S_2, $\mathscr{H}om_{\mathcal{O}_X}(\mathcal{O}_X(-C), \omega_X(A))$ is also S_2 by [AH11, Lem. 5.1.1]. Therefore we have $\mathscr{H}om_{\mathcal{O}_X}(\mathcal{O}_X(-C), \omega_X(A)) \cong \omega_X(C+A)$, since the two coincide outside a finite set of points and both are S_2. So there is a short exact sequence
$$0 \to \omega_X(A) \to \omega_X(C+A) \to \mathscr{E}xt^1_{\mathcal{O}_X}(\mathcal{O}_C, \omega_X(A)) \to 0. \tag{25}$$
By Remark A.1 we have an exact sequence
$$0 \to \mathcal{O}_X(-C-A) \to \mathcal{O}_X(-A) \to \mathcal{O}_C(-A) \to 0.$$
Applying $\mathscr{H}om_{\mathcal{O}_X}(\cdot, \omega_X)$ to this sequence, we get
$$\begin{aligned} 0 \to \mathscr{H}om_{\mathcal{O}_X}(\mathcal{O}_X(-A), \omega_X) &\to \mathscr{H}om_{\mathcal{O}_X}(\mathcal{O}_X(-C-A), \omega_X) \\ &\to \mathscr{E}xt^1_{\mathcal{O}_X}(\mathcal{O}_C(-A), \omega_X) \to \mathscr{E}xt^1_{\mathcal{O}_X}(\mathcal{O}_X(-A), \omega_X(A)). \end{aligned}$$
As before, by the S_2 property of the relevant sheaves, there are isomorphisms
$$\begin{aligned} \mathscr{H}om_{\mathcal{O}_X}(\mathcal{O}_X(-A), \omega_X) &\cong \omega_X(A), \\ \mathscr{H}om_{\mathcal{O}_X}(\mathcal{O}_X(-C-A), \omega_X) &\cong \omega_X(C+A). \end{aligned}$$
Also we have $\mathscr{E}xt^1_{\mathcal{O}_X}(\mathcal{O}_X(-A), \omega_X(A)) = 0$ by the proof of Lemma 3.3. So we obtain a short exact sequence
$$0 \to \omega_X(A) \to \omega_X(C+A) \to \mathscr{E}xt^1_{\mathcal{O}_X}(\mathcal{O}_C(-A), \omega_X) \to 0. \tag{26}$$
Comparing (25) and (26) gives $\mathscr{E}xt^1_{\mathcal{O}_X}(\mathcal{O}_C, \omega_X(A)) \cong \mathscr{E}xt^1_{\mathcal{O}_X}(\mathcal{O}_C(-A), \omega_X)$. Now, $\mathscr{H}om_{\mathcal{O}_X}(\mathcal{O}_C, \omega_X(A))$ and $\mathscr{H}om_{\mathcal{O}_X}(\mathcal{O}_C(-A), \omega_X)$ both being zero, the claim follows from the local-to-global-Ext-spectral-sequence. □

Proposition A.17 — *Let C and F be well-behaved curves on X and Y respectively such that $f_*F = C$ as Weil divisors. Then, for any $p \in X$, the vector spaces $R^1 f_*\mathcal{O}_Y(-F)_p$ and $(\omega_X(C)/f_*\omega_Y(F))_p$ are dual to each other.*

Proof. Let E be the reduced exceptional divisor over p. As in [Kol85, Lem. 3.3.3] we have, using Lemma A.16,
$$\begin{aligned} (\omega_X(C)/f_*\omega_Y(F))_p &= H^1_E(\omega_Y(F)) \\ &= \varinjlim_n \operatorname{Ext}^1_{\mathcal{O}_Y}(\mathcal{O}_{nE}, \omega_Y(F)) \cong \varinjlim_n \operatorname{Ext}^1_{\mathcal{O}_Y}(\mathcal{O}_{nE}(-F), \omega_Y) \end{aligned}$$
The surjection $\mathcal{O}_Y(-F)|_{nE} \twoheadrightarrow \mathcal{O}_{nE}(-F)$ has torsion kernel and thus by Serre duality
$$\operatorname{Ext}^1_{\mathcal{O}_Y}(\mathcal{O}_{nE}(-F), \omega_Y) = H^1(Y, \mathcal{O}_{nE}(-F))^\vee = H^1(Y, \mathcal{O}_Y(-F)|_{nE})^\vee.$$
Combining these equations we have by the theorem of formal functions ([Har77, Thm. III.11.1])
$$(\omega_X(C)/f_*\omega_Y(F))_p \cong \varprojlim_n H^1(Y, \mathcal{O}_Y(-F)|_{nE})^\vee = R^1 f_*\mathcal{O}_Y(-F)_p^\vee$$
which concludes the proof. □

A.6. **The hat transform.**

Proposition/Definition A.18 — *Let $B \subset X$ be a well-behaved curve. Then there exists a unique minimal integral almost Cartier divisor $\hat{B}_Y \subset Y$ such that $f_*\hat{B}_Y = B$ and for all exceptional divisors E of f*

$$\hat{B}_Y E \leq 0.$$

We call $\hat{B}_Y$ the hat transform of B with respect to f.

Proof. We can take a well-behaved very ample Cartier divisor H of X that contains B. Then $f^*H - (f^{-1})_*(H-B)$ contains the strict transform of B and has nonpositive intersection with any exceptional divisor E_i for any i. The existence follows.

Suppose $\hat{B}_1$ and $\hat{B}_2$ are two hat transforms of B under f. Write $\hat{B}_1 = \hat{B}_3 + A_1, \hat{B}_2 = \hat{B}_3 + A_2$, where A_1 and A_2 are two well-behaved effective divisors with no common irreducible components. Let E be a reduced and irreducible exceptional divisor of f. If $E \subset A_1$ then $E \not\subseteq A_2$, and $\hat{B}_3 E = (\hat{B}_2 - A_2)E \leq 0$; if $E \subset A_2$ then $E \not\subseteq A_1$, and $\hat{B}_3 E = (\hat{B}_1 - A_1)E \leq 0$. By the minimality of a hat transform we have $\hat{B}_1 = \hat{B}_2 = \hat{B}_3$. The uniqueness is proved. □

We start to gather some properties of the hat transform.

Lemma A.19 — *Let $\hat{B}_Y \subset Y$ be* the *hat transform of B. Then the following holds.*

(*i*) *$\hat{B}_Y - B_Y$ contains only exceptional curves of $f\colon Y \to X$.*
(*ii*) *Let $B_Y^* := f^*B_Y = B_Y + \Gamma^*$ be the numerical pullback of B_Y, so that $B_Y^* E_i = 0$ for any exceptional curve E_i of $f\colon Y \to X$. Then $\hat{B}_Y \geq B_Y^*$.*
(*iii*) *If $C \subset X$ is an effective Cartier divisor such that $B \leq C$ then $\hat{B} \leq f^*C$.*

In particular, if B is Cartier then $\hat{B}_Y = B_Y^*$.

Proof. Recall that $\hat{B}_Y$ is well-behaved.

For (i), if $\hat{B}_Y - B$ contains some curve A that is not exceptional then $(\hat{B}_Y - A)E_i \leq 0$ for any exceptional E_i, contradicting the minimality of $\hat{B}_Y$.

For (ii), note that $B_Y^* E = 0$ for any exceptional divisor E . So $(\hat{B}_Y - B_Y^*)E \leq 0$ for any exceptional curve E. Since the intersection form is negative definite on the exceptional divisors and $\hat{B}_Y - B_Y^*$ is supported only on exceptional divisors, we have $\hat{B}_Y \geq B_Y^*$.

For (iii), note that f^*C is an integral divisor, since C is Cartier. Moreover $f^*CE = 0$ for any exceptional curve E, and the strict transform B_Y of B is contained in f^*C. By the minimality of $\hat{B}_Y$, the inequality $\hat{B}_Y \leq f^*C$ follows. □

Remark A.20 — Note that since we used normalisation to define intersection numbers both $\hat{B}_Y$ and B_Y^* can behave unexpectedly: they might not contain all exceptional curves mapping to B. See also Remark 2.7.

Proposition A.21 — *In the situation above we have*

(*i*) $R^1 f_*\omega_Y(\hat{B}_Y) = 0$;
(*ii*) $R^1 f_*\omega_{\hat{B}_Y} = 0$;
(*iii*) $\chi(\omega_{\hat{B}_Y}) = \chi(f_*\omega_{\hat{B}_Y})$.

Proof. For (i) we look at the diagram (11) and consider the exact sequence

$$0 \to \eta_*\omega_{\bar{Y}}(\hat{B}_{\bar{Y}}) \to \omega_Y(\hat{B}_Y) \to \mathcal{Q}_Y \to 0$$

where $\mathcal{Q}_Y$ is supported on D_Y. Applying f_* we have

$$R^1 f_*\eta_*\omega_{\bar Y}(\hat B_{\bar Y}) \to R^1 f_*\omega_Y(\hat B_Y) \to R^1 f_*\mathcal{Q}_Y. \tag{27}$$

Since $f|_{D_Y}$ is finite, $R^1 f_*\mathcal{Q}_Y = 0$. On the other hand, using the Leray spectral sequence and finiteness of η and π, we have

$$\begin{aligned} R^1 f_*\eta_*\omega_{\bar Y}(\hat B_{\bar Y}) &= R^1(f\eta)_*\omega_{\bar Y}(\hat B_{\bar Y}) \\ &= R^1(\pi\bar f)_*\omega_{\bar Y}(\hat B_{\bar Y}) \\ &= \pi_* R^1\bar f_*\omega_{\bar Y}(\hat B_{\bar Y}). \end{aligned}$$

The argument for [CFHR99, Claim 4.3 (iv)] gives $R^1\bar f_*\omega_{\bar Y}(\hat B_{\bar Y}) = 0$ and hence $R^1 f_*\eta_*\omega_{\bar Y}(\hat B_{\bar Y}) = 0$. Now the vanishing of $R^1 f_*\omega_Y(\hat B_Y)$ follows from the exact sequence (27).

For (ii), we apply $\mathscr{H}om_{\mathcal{O}_Y}(\cdot, \omega_Y)$ to the short exact sequence

$$0 \to \mathcal{O}_Y(-\hat B_Y) \to \mathcal{O}_Y \to \mathcal{O}_{\hat B_Y} \to 0,$$

and get

$$0 \to \omega_Y \to \omega_Y(\hat B_Y) \to \omega_{\hat B_Y} \to 0.$$

Applying f_* to the above sequence and using (i) gives $R^1 f_*\omega_{\hat B_Y} \cong R^1 f_*\omega_Y(\hat B_Y) = 0$.

The last item is a direct consequence of (ii) and the Leray spectral sequence. $\square$

We now prove the main result of the section, an estimate for the change in arithmetic genus of the strict transform of the hat transform.

Proposition A.22 — *Let B be a well-behaved curve on X and $\hat B_{\bar Y} \subset \bar Y$ the strict transform of the hat transform of B. Then*

$$p_a(B) \le p_a(\hat B_{\bar Y}) + \frac{\hat B_{\bar Y} D_{\bar Y}}{2}.$$

Proof of Proposition A.22. The short exact sequence $0 \to \omega_X \to \omega_X(B) \to \omega_B \to 0$ fits into the diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & \mathcal{K}_B \arrow[d, hook] & \\
0 \arrow[r] & f_*\omega_Y \arrow[r] \arrow[d] & f_*\omega_Y(\hat B_Y) \arrow[r] \arrow[d] & f_*\omega_{\hat B_Y} \arrow[r] \arrow[d, "\exists"'] & 0 \\
0 \arrow[r] & \omega_X \arrow[r] \arrow[d] & \omega_X(B) \arrow[r] \arrow[d] & \omega_B \arrow[r] \arrow[d] & 0 \\
 & \omega_X/f_*\omega_Y \arrow[r] \arrow[d] & \omega(X)(B)/f_*\omega_Y(\hat B_Y) \arrow[r] \arrow[d] & \mathcal{Q}_B \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0. &
\end{tikzcd}$$

The first row is exact by Proposition A.21 and the rest of the diagram is exact by defining $\mathcal{K}_B$ (resp. $\mathcal{Q}_B$) to be the kernel (resp. cokernel) of $f_*\omega_{\hat B_Y} \to \omega_B$. The snake lemma together with the duality from Lemma A.17 gives

$$\dim_{\mathbb{C}} \mathcal{K}_B - \dim_{\mathbb{C}} \mathcal{Q}_B = h^0(R^1 f_*\mathcal{O}_Y) - h^0(R^1 f_*\mathcal{O}_Y(-\hat B_Y)).$$

Now we have

$$\begin{aligned}
p_a(\hat{B}_Y) &= 1 - \chi(\mathcal{O}_{\hat{B}_Y}) \\
&= 1 + \chi(\omega_{\hat{B}_Y}) \\
&= 1 + \chi(f_*\omega_{\hat{B}_Y}) \qquad \text{(by Prop. A.21}(iii)) \\
&= 1 + \chi(\omega_B) + \dim_{\mathbb{C}} \mathcal{K}_B - \dim_{\mathbb{C}} \mathcal{Q}_B \\
&= 1 + \chi(\omega_B) + h^0(R^1 f_*\mathcal{O}_Y) - h^0(R^1 f_*\mathcal{O}_Y(-\hat{B}_Y)) \\
&\geq p_a(B) - \frac{1}{2}\#\left\{ q \in \hat{B}_Y \cap D_Y \mid d_q(\hat{B}_Y) > 0 \right\} \qquad \text{(by Lem. A.15)} \\
&\geq p_a(B) - \frac{1}{2}\sum_{q \in Y_{\mathrm{sing}}} d_q(\hat{B}_Y)
\end{aligned}$$

Thus, using Proposition A.5(ii) for $\hat{B}_Y$, we get

$$\begin{aligned}
p_a(B) &\leq p_a(\hat{B}_Y) + \frac{1}{2}\sum_{q \in Y_{\mathrm{sing}}} d_q(\hat{B}_Y) \\
&= p_a(\hat{B}_{\bar{Y}}) + \sum_{q \in Y_{\mathrm{sing}}} n_q(\hat{B}_Y) + \frac{1}{2}\sum_{q \in Y_{\mathrm{sing}}} d_q(\hat{B}_Y) \\
&= p_a(\hat{B}_{\bar{Y}}) + \frac{\hat{B}_{\bar{Y}} D_{\bar{Y}}}{2}.
\end{aligned}$$

where the last equality is by Definition A.3. This concludes the proof. □

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